REMARKS ON PRODUCTS OF GENERALIZED TOPOLOGIES

R. SHEN*

Department of Mathematics, Nanjing University, Nanjing 210093, P.R. China;
Department of Mathematics, Taizhou Teachers’ College, Taizhou 225300, P.R. China
e-mail: src202120210163.com; rxshen@yahoo.cn

(Received October 30, 2008; accepted December 9, 2008)

Abstract. We study the relationship between the product and other basic operations (namely $\sigma$, $\pi$, $\alpha$ and $\beta$) of generalized topologies. Also we discuss the connectedness, generalized connectedness and compactness of products of generalized topologies. It is proved that the connectedness and compactness are preserved under the product of generalized topologies, which shows that the definition of product of generalized topologies is quite reasonable.

1. Introduction

The theory of generalized topological spaces, which was founded by Á. Császár, is one of the most important development of general topology in recent years. In [7], Á. Császár defined the product of generalized topologies and obtained some basic results. It is well-known that there are many kinds of products of topologies in the theory of general topology. Among them the most useful and important one is the Tychonoff product, on which we have many beautiful results. Naturally, we are interested in the question that whether these results still hold in generalized topological spaces. In this

*Supported in part by the NSFC (No. 10571151).

Key words and phrases: generalized topology, product, connectedness, compactness.

2000 Mathematics Subject Classification: 54A05; 54D15.
paper, we study the relationship between the product and other basic operations (namely $\sigma$, $\pi$, $\alpha$ and $\beta$) of generalized topologies. Also we discuss the connectedness, generalized connectedness and compactness of products of generalized topologies. It is proved that the connectedness and compactness are preserved under the product of generalized topologies, which shows that the definition of product of generalized topologies is quite reasonable.

We recall some basic definitions and notations. Let $X$ be a set, and denote $\exp X$ the power set of $X$. We call a class $\mu \subseteq \exp X$ a \textit{generalized topology} [3] (briefly GT) if $\emptyset \in \mu$ and any union of elements of $\mu$ belongs to $\mu$. A set with a GT is said to be a \textit{generalized topological space} (briefly GTS). For a GTS $(X, \mu)$, the elements of $\mu$ are called $\mu$-open sets and the complements of $\mu$-open sets are called $\mu$-closed sets. Let $M_{\mu}$ denote the union of all elements of $\mu$. We say $\mu$ is \textit{strong} [5] if $M_{\mu} = X$. For $A \subseteq X$, we denote by $cA$ the intersection of all $\mu$-closed sets containing $A$ and by $iA$ the union of all $\mu$-open sets contained in $A$. Then we have $iA = iA$, $ccA = cA$ and $iA = X - c(X - A)$. We call $c: \exp X \to \exp X$ and $i: \exp X \to \exp X$ the closure operation and the interior operation of $(X, \mu)$, respectively. A set $A \subseteq X$ is said to be $\mu$-semi-open (resp. $\mu$-preopen, $\mu$-$\alpha$-open, $\mu$-$\beta$-open) [6] if $A \subset ciA$ (resp. $A \subset icA$, $A \subset cciA$, $A \subset cciA$). We denote by $\sigma(\mu)$ (resp. $\pi(\mu)$, $\alpha(\mu)$, $\beta(\mu)$) the class of all $\mu$-semi-open sets (resp. $\mu$-preopen sets, $\mu$-$\alpha$-open sets, $\mu$-$\beta$-open sets). Obviously $\mu \subseteq \alpha(\mu) \subseteq \sigma(\mu) \subseteq \beta(\mu)$ and $\alpha(\mu) \subseteq \pi(\mu) \subseteq \beta(\mu)$. Further we have $M_{\alpha(\mu)} = M_{c(\mu)} = M_{\mu}$ and $M_{\sigma(\mu)} = M_{\beta(\mu)} = X$.

Let $K \neq \emptyset$ be an index set and $(X_k, \mu_k)$ ($k \in K$) a class of GTS’s. $X = \prod_{k \in K} X_k$ is the Cartesian product of the sets $X_k$. Let us consider all sets of the form $\prod_{k \in K} B_k$ where $B_k \in \mu_k$ and, with the exception of a finite number of indices $k$, $B_k = M_{\mu_k}$. We denote by $\mathcal{B}$ the collection of all these sets. We call $\mu = \mu(\mathcal{B})$ having $\mathcal{B}$ as a base the product [7] of the GT’s $\mu_k$ and denote it by $\mathbf{P}_{k \in K} \mu_k$. The GTS $(X, \mu)$ is called the product of the GTS’s $(X_k, \mu_k)$.

Consider two GTS’s $(X_k, \mu_k)$ and $(X', \mu')$. A mapping $f: X \to X'$ is \textit{$(\mu, \mu')$-continuous} [3] if $U \in \mu'$ implies that $f^{-1}(U) \in \mu$. Throughout this paper, all mappings are assumed to be onto.

2. The product and other basic operations of GT’s

Let $K \neq \emptyset$ be an index set and $(X, \mu)$ the product of the GT’s $(X_k, \mu_k)$, $k \in K$. Denote $M_k = \bigcup \mu_k$ and $M = \bigcup \mu$. By [7, Lemma 2.6], $M = \prod_{k \in K} M_k$. We denote by $p_k$ the projection $X \to X_k$ and $x_k = p_k(x)$ for each $x \in X$. Let us write $i = i_\mu$, $c = c_\mu$, $i_k = i_{\mu_k}$, $c_k = c_{\mu_k}$.

\textbf{Proposition 2.1.} Let $A = \prod_{k \in K} A_k \subseteq \prod_{k \in K} X_k$ and $K_0$ be a finite subset of $K$. If $A_k \in \{M_k, X_k\}$ for each $k \in K - K_0$, then $iA = \prod_{k \in K} i_k A_k$. 

\textit{Acta Mathematica Hungarica} 124, 2009