INDUCED GENERALIZED $S$-SPACE-FORM STRUCTURES ON SUBMANIFOLDS

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Abstract. We study non-anti-invariant slant submanifolds of generalized $S$-space-forms with two structure vector fields in order to know if they inherit the ambient structure. In this context, we focus on totally geodesic, totally umbilical, totally $f$-geodesic and totally $f$-umbilical non-anti-invariant slant submanifolds and obtain some obstructions. Moreover, we present some new interesting examples of generalized $S$-space-forms.

1. Introduction

It is an interesting problem to analyze what kind of Riemannian manifolds may be determined by special pointwise expressions for their curvatures. For instance, it is well known that the sectional curvatures of a Riemannian manifold determine the curvature tensor field completely. So, if $(M, g)$ is a connected Riemannian manifold with dimension greater than 2 and its curvature tensor field $R$ has the pointwise expression

$$R(X, Y)Z = \lambda \{ g(X, Z)Y - g(Y, Z)X \},$$

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where $\lambda$ is a differentiable function on $M$, then $M$ is a space of constant sectional curvature, that is, a real-space-form and $\lambda$ is a constant function.

Furthermore, when the manifold is equipped with some additional structure, it is sometimes possible to obtain conclusions from the special form of the curvature tensor field for this structure too. Thus, for almost-Hermitian manifolds, F. Tricerri and L. Vanhecke [18] introduced \textit{generalized complex-space-forms} and, for almost contact metric manifolds, P. Alegre, D. E. Blair and A. Carriazo [1] defined and studied \textit{generalized Sasakian-space-forms}.

More generally, K. Yano [19] introduced the notion of $f$-structure on a $2m + s$-dimensional manifold as a tensor field $f$ of type $(1, 1)$ and rank $2m$ satisfying $f^3 + f = 0$. Almost complex ($s = 0$) and almost contact ($s = 1$) structures are well-known examples of $f$-structures. In this context, D. E. Blair [2] defined $K$-manifolds (and particular cases of $S$-manifolds and $C$-manifolds) as the analogue of Kaehlerian manifolds in the almost complex geometry and of quasi-Sasakian manifolds in the almost contact geometry and he showed that the curvature of either $S$-manifolds or $C$-manifolds is completely determined by their $f$-sectional curvatures. Later, M. Kobayashi and S. Tsuchiya [15] got expressions for the curvature tensor field of $S$-manifolds and $C$-manifolds when their $f$-sectional curvature is constant depending on such a constant. Nice examples of $S$-manifolds with constant $f$-sectional curvature can be found in [2, 14]. In particular, Blair proved in [2] that certain principal toroidal bundles over complex-space-forms are $S$-manifolds with constant $f$-sectional curvature and he introduced a generalization of the Hopf fibration as a canonical example of such manifolds playing the role of complex projective space in Kaehler geometry and the odd-dimensional sphere in Sasakian geometry.

For these reasons, the authors and A. M. Fuentes introduced a notion of \textit{generalized $S$-space-form} on metric $f$-manifolds (see [6]) and they limited the research to the case $s = 2$ which appeared in the study of hypersurfaces in almost contact manifolds [3, 13], giving some non-trivial examples. But, as in any new subject, one of the most important aspect is the search for more examples. In this sense, a natural question arises: when does a submanifold of a generalized $S$-space-form inherit the ambient structure? Of course, two things have to be inherited from the ambient space. First, the $f$-structure and so, the structure vector fields have to be tangent to the submanifold. The trivial situation is that of invariant submanifolds. Thus, a natural selection seems to be that of non-anti-invariant slant submanifolds (for some general background on the theory of slant submanifolds, the survey paper [5] can be consulted).

On the other hand, in order to be a generalized $S$-space-form, the curvature tensor of the submanifold has to be written in a special way and the Gauss equation is very useful. So, we must somehow control the second fundamental form of the immersion to obtain a suitable expression for such a