THE ULTIMATE CONDITION TO GENERALIZE MONOTONICITY FOR ABEL’S AND DIRICHLET’S CRITERIA

R. J. LE¹,∗ and S. P. ZHOU²,†

¹Faculty of Science, Ningbo University, Ningbo 315211, China
e-mail: leruijun@163.com

²Institute of Mathematics, Zhejiang Sci-Tech University, Hangzhou 310018, China
e-mail: songping.zhou@163.com

(Received November 8, 2013; revised November 10, 2013; accepted November 11, 2013)

Abstract. It could be very interesting to generalize the monotonicity condition of two classical Abel’s and Dirichlet’s criteria for the convergence of numerical series. In the present note, we give this problem a brief complete answer. Especially, we indicate that the positivity in the rest bounded variation concept by Leindler can be dropped.

1. Introduction

In classical analysis, an interesting problem is the generalization of monotonicity, especially for numerical sequences. For instance, since Chaundy and Jolliffe [1] proved that if \( \{a_n\} \) is a decreasing nonnegative sequence, then \( \sum_{n=0}^{\infty} a_n \sin nx \) uniformly converges if and only if \( \lim_{n \to \infty} n a_n = 0 \), many scholars have produced various quasimonotonicity conditions (quasimonotonicity, \( O \)-regularly varying quasimonotonicity) and various bounded variation conditions (rest bounded variation, group bounded variation, mean value bounded variation, etc.) to generalize (decreasing) monotonicity while keeping the Chaundy–Jolliffe’s theorem true. The reference [4] or [5] gives the various definitions of those sequences and a survey of the history.

The following condition generalizing (decreasing) monotonicity is called “rest bounded variation” condition:

∗ Supported by Scientific Research Fund of Zhejiang Provincial Education Department (No. ZX2012000178), K. C. Wong Magna Fund in Ningbo University
† Corresponding author.

Key words and phrases: numerical series, rest bounded variation, convergence, Abel’s and Dirichlet’s criteria.

Mathematics Subject Classification: 40A05.
DEFINITION 1.1. A real null sequence $A := \{a_n\}_{n=1}^{\infty}$ is called “rest bounded variation”, if for all $m \geq 1$,

$$\sum_{n=m}^{\infty} |\Delta a_n| := \sum_{n=m}^{\infty} |a_n - a_{n+1}| \leq M(A)|a_m|$$

holds, where $M(A)$ is a positive constant depending on the sequence $A$ only.

For nonnegative sequences, this condition is raised by Leindler [3].

We know, monotonicity is also applied to some very important basic results in analysis. For instance, among the criteria for convergence or uniform convergence of series, two well-known of those are Abel’s criterion and Dirichlet’s criterion as follows:

**Abel’s Criterion.** If $\{a_n\}$ is monotone and bounded, $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n b_n$ converges.

**Dirichlet’s Criterion.** If $\{a_n\}$ decreases and tends to zero, $\sum_{k=1}^{n} b_k$ is bounded, then $\sum_{n=1}^{\infty} a_n b_n$ converges.

It would be very interesting to generalize the monotonicity condition of those two classical results. After investigation, [2] found that monotonicity can be generalized to rest bounded variation condition in these two criteria, while unfortunately, quasimonotonicity is not the right choice.

In the present note, we give this problem a brief complete answer. Especially, we indicate that the positivity in the rest bounded variation concept by Leindler can be dropped.

Throughout the note, we always use $M$ to stand for a positive constant that may not be necessarily the same at each occurrence, sometimes also use $O(1)$ to indicate the same meaning.

2. Results and proofs

Let $\{R(n)\}$ be an increasing sequence with $R(n + 1)/R(n) = O(1)$. Assume that a real sequence $\{a_n\}$ satisfies

$$\lim_{n \to \infty} a_n = 0$$

and

$$\sum_{k=n}^{\infty} \left| \frac{\Delta a_k}{R(k)} \right| \leq M \frac{|a_n|}{R(n)}.$$