ON THE PARALLEL SUM OF POSITIVE OPERATORS, FORMS, AND FUNCTIONALS

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Abstract. The parallel sum \( A : B \) of two bounded positive linear operators \( A, B \) on a Hilbert space \( H \) is defined to be the positive operator having the quadratic form

\[
\inf \{(A(x - y) \mid x - y) + (By \mid y) \mid y \in H\}
\]

for fixed \( x \in H \). The purpose of this paper is to provide a factorization of the parallel sum of the form \( J_A P J_A^* \) where \( J_A \) is the embedding operator of an auxiliary Hilbert space associated with \( A \) and \( B \), and \( P \) is an orthogonal projection onto a certain linear subspace of that Hilbert space. We give similar factorizations of the parallel sum of nonnegative Hermitian forms, positive operators of a complex Banach space \( E \) into its topological anti-dual \( E' \), and of representable positive functionals on a \(*\)-algebra.

1. Introduction

Let \( H \) be a complex Hilbert space with inner product \((\cdot \mid \cdot)\) and denote by \( \mathcal{B}(H) \) the \( C^* \)-algebra of all continuous linear operators acting on \( H \). An operator \( A \in \mathcal{B}(H) \) is called positive, as usual, if its quadratic form is nonnegative, that is,

\[
(Ax \mid x) \geq 0, \quad x \in H.
\]

If we are given another positive operator \( B \in \mathcal{B}(H) \) then the parallel sum of \( A \) and \( B \) can be defined being the (unique) positive operator \( A : B \) possessing the quadratic form

\[
(A : B)x \mid x = \inf_{y \in H} \{(A(x - y) \mid x - y) + (By \mid y)\},
\]

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cf. Ando [3]. The parallel sum has been introduced first by Anderson and Duffin [2] for positive definite matrices, i.e., for positive operators on a finite dimensional Hilbert space by setting $A : B = A(A + B)^{-1}B$. The concept of parallel sum of (not necessarily invertible) positive operators on arbitrary Hilbert spaces is due to Anderson [1]. This concept has been studied by plenty of authors (see eg. [3,4,9]) and has found a wide range of applications. In the present work we give a new construction of the parallel sum. Namely, we provide $A : B$ via factorization of the form $A : B = J_A P J_A^*$ where $P$ is an orthogonal projection of an auxiliary Hilbert space and $J_A$ is a bounded linear operator from that space into $H$. Our method is based on the observation that the square root of formula (1.1) can be regarded as the distance between $Ax$ and $\{(Ay, By) \mid y \in H\}$ in an appropriate metric. In this section we also provide some further factorizations of the parallel sum. One of these enables us to characterize the range of $(A - (A : B))^{1/2}$.

Ando [3] applied the parallel sum as a powerful tool by investigating Lebesgue type decompositions of positive operators. His definition (1.1) of the parallel sum however makes only use of the quadratic forms of the operators under consideration (the reader is referred to [16] or [18] for a purely operator theoretic approach to the Lebesgue decomposition). This observation made it possible to introduce the parallel sum of two nonnegative Hermitian forms. In fact, if $t$ and $w$ are forms on a complex vector space $\mathcal{D}$ then

$$ (t : w)(x, x) := \inf_{y \in H} \{t(x - y, x - y) + w(y, y)\}, \quad x \in \mathcal{D}, $$

fulfills the parallelogram law and hence, according to the Jordan–Neumann theorem, $t : w$ is a form itself, see [5, Proposition 2.2]. In Section 3 we provide $t : w$ in a different way: it is shown to be the quadratic form of an appropriately chosen positive operator.

A natural generalization of positive operators of Hilbert spaces to Banach space setting is in considering linear operators of a Banach space $E$ to the topological anti-dual $E'$, satisfying $\langle Ax, x \rangle \geq 0$. In Section 4 we introduce the parallel sum of positive operators in this setting.

If $f$ and $g$ are positive functionals on a $*$-algebra $\mathcal{A}$ then, of course, $t_f(a, b) := f(b^*a)$ and $t_g(a, b) := g(b^*a)$ define nonnegative Hermitian forms on $\mathcal{A}$, thus the parallel sum $t_f : t_g$ of these forms can be considered. Nevertheless, it is not clear if $t_f : t_g = t_h$ holds with some positive functional $h$. In Section 5, based on the Gelfand–Neumark–Segal construction we shall show that this is the case when considering representable positive functionals. Furthermore, we give a characterization of singularity of representable positive functionals in terms of the parallel sum.