SOLUTION OF A PROBLEM OF Z. DARÓCZY, J. JARCZYK AND W. JARCZYK AND GENERALIZATION OF A THEOREM OF R. GER AND T. KOCHANEK

M. BALCEROWSKI
ul. 1 Maja 56/80, 41-200 Sosnowiec, Poland
e-mail: mbalcerowski@math.us.edu.pl

(Received June 7, 2017; revised December 13, 2017; accepted December 18, 2017)

Abstract. We generalize a theorem proved by R. Ger and T. Kochanek [4]. Hence we obtain the solution of a problem posed by Z. Daróczy, J. Jarczyk and W. Jarczyk [2].

We start with recalling fundamental notions. If \( I \subset \mathbb{R} \) is an interval and \( M: \mathbb{R}^2 \to \mathbb{R} \) is a function, then \( M \) is called a mean provided

\[
\min\{x, y\} \leq M(x, y) \leq \max\{x, y\} \quad \text{for every} \quad x, y \in I.
\]

Let \( I \subset \mathbb{R} \) be an interval and let \( M: \mathbb{R}^2 \to \mathbb{R} \) be a mean. The mean \( M \) is called strict, if \( \min\{x, y\} < M(x, y) < \max\{x, y\} \) for every \( x, y \in I \) such that \( x \neq y \); and quasi-arithmetic, if there exists a continuous and strictly monotonic function \( g: I \to \mathbb{R} \) such that

\[
M(x, y) = g^{-1}\left(\frac{g(x) + g(y)}{2}\right)
\]

for every \( x, y \in I \). In this situation we say that \( g \) generates \( M \).

**Remark 1.** If \( I \subset \mathbb{R} \) is an interval and \( g: I \to \mathbb{R} \) is continuous and strictly monotonic, then the function \( M: I^2 \to I \), given by (1), is a continuous mean strictly increasing with respect to each variable. Moreover, if \( h \) is a generator of a quasi-arithmetic mean, then the function \( -h \) generates the same mean. Therefore every quasi-arithmetic mean has a continuous strictly increasing generator.

**Key words and phrases:** functional equation, mean, quasi-arithmetic mean, strict mean.

**Mathematics Subject Classification:** primary 26E60, 39B12.
Now we present the origin of the title problem. R. Ger and T. Kochanek [4] proved the following result.

**Theorem GK.** Let $I, J \subset \mathbb{R}$ be intervals and let $M: I^2 \to I$ and $K: J^2 \to J$ be means continuous with respect to each variable and strictly increasing with respect to each variable. Assume that the equation

$$(2) \quad f(M(x, y)) = K(f(x), f(y))$$

has a non-constant solution $f: I \to J$. Then the following conditions hold:

(a) if the mean $K$ is quasi-arithmetic, then $M$ is quasi-arithmetic;
(b) if the mean $M$ is quasi-arithmetic, then $K$ is quasi-arithmetic on $(\inf f(I), \sup f(I))^2$.

During the 52th International Symposium of Functional Equations [1] it was discussed problems which were related to Theorem GK. The problem below was posed by Z. Daróczy.

**Problem 1.** Let $I \subset \mathbb{R}$ be an interval and let $M: I^2 \to I$ be a mean. Is it true that if $M$ is not quasi-arithmetic, then every solution $f: I \to \mathbb{R}$ of the equation

$$(3) \quad f(M(x, y)) = \frac{f(x) + f(y)}{2}$$

is constant?

Z. Daróczy, J. Jarczyk and W. Jarczyk answered this problem negatively (still during the Symposium) presenting an example of a continuous mean $M$ increasing with respect to each variable which satisfies equation (3) with a non-constant function $f$. Next they asked what is the answer to Problem 1 under the additional assumption that the mean $M$ is strict. In [2] they proved that the answer is still negative and they posed the following problem: what is the answer to Problem 1 under the additional assumption that the mean $M$ is strict and continuous?

Theorem 1 below answers this question. This theorem also generalizes Theorem GK.

**Theorem 1.** Let $I, J \subset \mathbb{R}$ be intervals, $M: I^2 \to I$ be a strict mean continuous with respect to each variable and let $K: J^2 \to J$ be a quasi-arithmetic mean. If equation (2) has a non-constant solution $f: I \to J$, then the mean $M$ is quasi-arithmetic.

To prove Theorem 1 we need Proposition 1 below.

**Proposition 1.** Let $I \subset \mathbb{R}$ be an interval, $J$ be a set, $f: I \to J$ be a non-constant function and let $K: J^2 \to J$ be a function. Assume that for every $x \in I$ the functions $K(f(x), \cdot)|_{f(I)}$ and $K(\cdot, f(x))|_{f(I)}$ are one-to-one.