Interpolation of Besov $B^{\sigma q}_{p\tau}$ and Lizorkin–Triebel $F^{\sigma q}_{p\tau}$ spaces

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Received April 23, 2008.

Abstract. An interpolation method is introduced for anisotropic spaces which generalizes the method by D. L. Fernandez [4]. By means of this method, interpolation properties of Besov $B^{\sigma q}_{p\tau}$ and Lizorkin–Triebel $F^{\sigma q}_{p\tau}$ spaces are investigated. Among others, the completeness of the scale of these spaces is proved with respect to the considered interpolation method.

Introduction

The first interpolation theorems for Sobolev and Besov spaces have appeared in Peetre’s work [9]. In particular, he showed that for $1 \leq p \leq \infty$, $\alpha_1 \in \mathbb{N}$, $0 < \theta < 1$ and $1 \leq q \leq \infty$ the equality

$$(L_p, W^{\alpha_1}_p)_{\theta q} = B^{\alpha q}_p, \quad \alpha = \theta \alpha_1$$

is correct.

Let’s note Lions and Peetre’s work [7], in which the following equality

$$(B^{\alpha_0 q_0}_{p_0}, B^{\alpha_1 q_1}_{p_1})_{\theta q} = B^{\alpha q}_p,$$

is studied for

$$1 \leq p_0 \neq p_1 \leq \infty, \quad 0 < \alpha_0 \neq \alpha_1 < \infty, \quad 1 \leq q_0 \neq q_1 \leq \infty$$

and

$$0 < \theta < 1, \quad 1 \leq q \leq \infty, \quad \text{where} \quad \alpha = (1 - \theta)\alpha_0 + \theta \alpha_1.$$
is correct only in case
\[ q = p \quad \text{and} \quad 1/p = (1 - \theta)/p_0 + \theta/p_1 = (1 - \theta)/q_0 + \theta/q_1 \]
(so-called diagonal case). In a nondiagonal case, the result of interpolation of Besov spaces is not described by spaces from same scale.

Further study of interpolation properties of Besov and Lizorkin–Triebel spaces, concerning the classical complex and real methods, the works of many authors were devoted, with review it is possible to familiarize in the monographies of Berg and Levstrem [3], Triebel [10].

In Krepkogorski’s works [5], [6], interpolation properties of Besov and Lizorkin–Triebel spaces in a nondiagonal case and concerning Sparr method were studied as follows. By Ascritova’s and others [1]: interpolation properties of Besov and Lizorkin–Triebel spaces concerning Sparr method, by the present authors [2]: interpolation properties of Besov spaces concerning a method of multiparameter interpolation.

The given work is devoted to the study of interpolation properties of Besov and Lizorkin–Triebel type spaces concerning interpolation method for anisotropic spaces (see in [8]).

In particular, the following theorem was proved.

**Theorem.** Let \(-\infty < \sigma_0 \neq \sigma_1 < \infty, 1 < p_0 \neq p_1 < \infty, 1 < q_0, q_1 < \infty\) and \(1 \leq \tau_0, \tau_1 \leq \infty\). Then for \(0 < \theta_1, \theta_2 < 1, 1 \leq \tau_1, \tau_2 \leq \infty\) the following equalities are correct:

a) at \(\star_1 = (1, 2)\)

\[
\begin{align*}
(B_{p_1 \tau_1}^{\sigma_1} q_2 (T^n); \varepsilon_1, \varepsilon_2 = 0, 1)_{(\theta_1, \theta_2), (r_1, r_2) \tau_1} & = B_{p_1 \tau_1}^{\sigma r_2} (T^n), \\
(F_{p_1 \tau_1}^{\sigma_1} q_2 (T^n); \varepsilon_1, \varepsilon_2 = 0, 1)_{(\theta_1, \theta_2), (r_1, r_2) \tau_1} & = F_{p_1 \tau_1}^{\sigma r_2} (T^n),
\end{align*}
\]

b) at \(\star_2 = (2, 1)\)

\[
\begin{align*}
(B_{p_1 \tau_1}^{\sigma_1} q_2 (T^n); \varepsilon_1, \varepsilon_2 = 0, 1)_{(\theta_1, \theta_2), (r_1, r_2) \tau_2} & = F_{p_1 \tau_1}^{\sigma r_2} (T^n), \\
(F_{p_1 \tau_1}^{\sigma_1} q_2 (T^n); \varepsilon_1, \varepsilon_2 = 0, 1)_{(\theta_1, \theta_2), (r_1, r_2) \tau_2} & = F_{p_1 \tau_1}^{\sigma r_2} (T^n),
\end{align*}
\]

where
\[
\sigma = (1 - \theta_2)\sigma_0 + \theta_2\sigma_1, \quad 1/p = (1 - \theta_1)/p_0 + \theta_1/p_1.
\]

It is worth noting that, in contrast to the real valuable method, the scale of Besov and Lizorkin–Triebel type spaces is closed in the case of the method given by us; so, it behaves like a complete scale of spaces.