On Hardy-type integral inequalities*

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Abstract The Hardy integral inequality is one of the most important inequalities in analysis. The present paper establishes some new Copson-Pachpatte (C-P) type inequalities, which are the generalizations of the Hardy integral inequalities on binary functions.

Key words Hardy inequality, Hölder inequality, Copson inequality, Izumi inequality, Pachpatte inequality

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1 Introduction

The well-known Hardy’s integral inequality is as follows:

**Theorem 1.1** If $p > 1$, $f(x) \geq 0$ for $0 < x < \infty$, and $F(x) = \frac{1}{x} \int_0^x f(t)dt$, then

$$
\int_0^\infty F(x)^p dx < \left( \frac{p}{p-1} \right)^p \int_0^\infty f(x)^p dx
$$

(1)

unless $f \equiv 0$. The constant is the best possible.

Theorem 1.1 was first proved by Hardy [1] in an attempt to give a simple proof of Hilbert’s double series theorem (see [2]). One of the best known and interesting generalization of the inequality (1) was given by Hardy himself in [3]. It can be stated as the following two theorems:

**Theorem 1.2** If $p > 1$, $m > 1$, $f(x) \geq 0$ for $0 < x < \infty$, and

$$
F(x) = \int_0^x f(t)dt,
$$

then

$$
\int_0^\infty x^{-m} F(x)^p dx < \left( \frac{p}{m-1} \right)^p \int_0^\infty x^{p-m} f(x)^p dx
$$

(2)

unless $f \equiv 0$. The constant is the best possible.

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**Theorem 1.3** If $p > 1$, $m < 1$, $f(x) \geq 0$ for $0 < x < \infty$, and

$$
F(x) = \int_x^\infty f(t)dt,
$$
then

$$
\int_0^\infty x^{-m}F(x)^pdx < \left(\frac{p}{1-m}\right)^p \int_0^\infty x^{p-m}f(x)^pdx
$$

(3)

unless $f \equiv 0$. The constant is the best possible.

The inequalities given in (1) and (2) are called the Hardy inequalities which lead to many papers dealing with the alternative proofs, various generalizations, and numerous variants and applications in analysis (see [4–14]). In particular, Pachpatte [4] and Copson [15] established some generalizations of Hardy inequalities (1) and (2), respectively. Following these, in this paper, we establish new Copson-Pachpatte (C-P) type inequalities. These results provide some new estimates to this type of inequalities, and in the special cases, yield some of the recent results.

2 Statement of results

To simplify our statement, we call the four binary functions $f$, $h$, $w$, and $r$ satisfying the C-P condition on $(0, \infty) \times (0, \infty)$ if

(i) $f(x, y)$ is a nonnegative and integrable function on $(0, \infty) \times (0, \infty)$;
(ii) $h(x, y)$ is a positive continuous function on $(0, \infty) \times (0, \infty)$;
(iii) $w(x, y)$ and $r(x, y)$ are positive and absolutely continuous functions on $(0, \infty) \times (0, \infty)$;
(iv) for almost all $(x, y) \in (0, \infty) \times (0, \infty)$, there is a positive constant $\alpha$ such that

$$
1 - \frac{1}{m-1} \frac{H(x, y)}{h(x, y)} \frac{1}{w(x, y)} \frac{\partial w(x, y)}{\partial x} + \frac{p}{m-1} \frac{H(x, y)}{h(x, y)} \frac{1}{r(x, y)} \frac{\partial r(x, y)}{\partial x} \geq \frac{1}{\alpha},
$$

where $H(x, y) = \int_0^x \int_0^y h(s, t)dsdt$.

Our main results are the following three theorems:

**Theorem 2.1** Let $p > 1$ and $m > 1$ be constants, the four binary functions $f$, $h$, $w$, and $r$ satisfy the C-P condition on $(0, \infty) \times (0, \infty)$, and

$$
F(x, y) = \frac{1}{r(x, y)} \int_0^x \int_0^y r(s, t)h(s, t)f(s, t)dsdt
$$

for $(x, y) \in (0, \infty) \times (0, \infty)$. Then,

$$
\int_0^\infty \int_0^\infty w(x, y)H(x, y)^{-m}v(x, y)F(x, y)^pdxdy
\leq \left(\alpha \left(\frac{p}{m-1}\right)\right)^p \int_0^\infty \int_0^\infty w(x, y)H(x, y)^{p-m}h(x, y)G(x, y)^pdxdy,
$$

(4)

where

$$
v(x, y) = \int_0^y h(x, t)dt, \quad G(x, y) = \frac{1}{r(x, y)} \int_0^y r(x, t)h(x, t)f(x, t)dt.
$$

The equality holds if $f(x, y) \equiv 0$. 

**Theorem 2.2**