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The Eilenberg-Moore Category and a Beck-type Theorem for a Morita Context

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Abstract The Eilenberg-Moore constructions and a Beck-type theorem for pairs of monads are described. More specifically, a notion of a *Morita context* comprising of two monads, two bialgebra functors and two connecting maps is introduced. It is shown that in many cases equivalences between categories of algebras are induced by such Morita contexts. The Eilenberg-Moore category of representations of a Morita context is constructed. This construction allows one to associate two pairs of adjoint functors with right adjoint functors having a common domain or a *double adjunction* to a Morita context. It is shown that, conversely, every Morita context arises from a double adjunction. The comparison functor between the domain of right adjoint functors in a double adjunction and the Eilenberg-Moore category of the associated Morita context is defined. The sufficient and necessary conditions for this comparison functor to be an equivalence (or for the *moritability* of a pair of functors with a common domain) are derived.

Keywords Monad · Adjoint functor · Morita context · Eilenberg-Moore category

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1 Introduction

In the last two decades, Hopf-Galois theory went through a series of generalisations, ultimately leading to the theory of Galois comodules over corings, which elucidates its relation with well-known Beck's monadicity theorem. This evolution has revived the interest among Hopf algebraists in the theory of (co)monads. Several aspects of Hopf(-Galois) theory have been reformulated in the framework of (co)monads so that further clarification of the underlying categorical mechanisms of the theory has been achieved. This has led in particular to two different approaches to the definition of a categorical (functorial) notion of a *herd* or *pre-torsor*, appearing almost simultaneously in [4] and [3]. The motivation for the present paper was to study the connection between these two approaches in more detail.

The definition of a pre-trosor in [3] takes a pair of adjunctions (with coinciding codomain category for the left adjoints) as a starting point. The aim of this paper is to show that there is a close relationship between pairs of adjunctions and functorial Morita contexts similar to the correspondence between single adjunctions and monads. In this sense the results presented here can be interpreted as a 'two-dimensional' version of the latter correspondence. A key feature of this work is that it links aspects of the theory that are of more algebraic nature (Morita contexts) with aspects that are of more categorical nature (Beck's theorem). More precisely, we prove a version of Beck's theorem on precise monadicity in this 'two-dimensional' setting and provide a categorical (monadic) version of classical Morita theory.

These are the main results and the organisation of the paper. In Section 2 we recall from [3] the definition of the category of double adjunctions on categories $\mathcal{X}$ and $\mathcal{Y}$, $\mathsf{Adj}(\mathcal{X}, \mathcal{Y})$, and introduce the category $\mathsf{Mor}(\mathcal{X}, \mathcal{Y})$ of (functorial) Morita contexts.

In Section 3 we describe functors connecting categories of double adjunctions and Morita contexts. More precisely we compare categories $\mathsf{Adj}(\mathcal{X}, \mathcal{Y})$ and $\mathsf{Mor}(\mathcal{X}, \mathcal{Y})$. First we define a functor $\Upsilon : \mathsf{Adj}(\mathcal{X}, \mathcal{Y}) \to \mathsf{Mor}(\mathcal{X}, \mathcal{Y})$. To construct a functor in the converse direction, to each Morita context $\mathbb{T}$ we associate its *Eilenberg-Moore category* $(\mathcal{X}, \mathcal{Y})^{\mathbb{T}}$. This is very reminiscent of the classical Eilenberg-Moore construction of algebras of a monad (recalled in Section 2.1), and, in a way, is based on doubling of the latter. Objects in $(\mathcal{X}, \mathcal{Y})^{\mathbb{T}}$ are two algebras, one for each monad in $\mathbb{T}$, together with two connecting morphisms. Once $(\mathcal{X}, \mathcal{Y})^{\mathbb{T}}$ is defined, two adjunctions, one between $(\mathcal{X}, \mathcal{Y})^{\mathbb{T}}$ and $\mathcal{X}$ the other between $(\mathcal{X}, \mathcal{Y})^{\mathbb{T}}$ and $\mathcal{Y}$, are constructed. This construction yields a functor $\Gamma : \mathsf{Mor}(\mathcal{X}, \mathcal{Y}) \to \mathsf{Adj}(\mathcal{X}, \mathcal{Y})$. Next it is shown that the functors (Γ, Υ) form an adjoint pair, and that Γ is a full and faithful functor. The counit of this adjunction is given by a *comparison functor K* which compares the common category $\mathcal{Z}$ in a double adjunction $\mathfrak{Z}$ with the Eilenberg-Moore category of the associated Morita context $\mathbb{T} = \Upsilon(\mathfrak{Z})$. A necessary and sufficent condition for the comparison functor to be an equivalence are derived. This is closely related to the existence of colimits of diagrams of certain type in $\mathcal{Z}$ and is a Morita–double adjunction version of the classical Beck theorem (on precise monadicity).

In Section 4 we analyse which objects of $\mathsf{Mor}(\mathcal{X}, \mathcal{Y})$ describe equivalences between categories of algebras of monads. It is also proven that large classes of equivalences between categories of algebras are induced by Morita contexts.

In Section 5 examples and special cases of the theory developed in preceding sections are given. In particular, it is shown how the main results of Section 3 can