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On Certain 2-Categories Admitting Localisation by Bicategories of Fractions

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Abstract Pronk's theorem on bicategories of fractions is applied, in almost all cases in the literature, to 2-categories of geometrically presentable stacks on a 1-site. We give an proof that subsumes all previous such results and which is purely 2-categorical in nature, ignoring the nature of the objects involved. The proof holds for 2-categories that are not (2,1)-categories, and we give conditions for local essential smallness. This is the published version of arXiv:1402.7108.

Keywords 2-categories · Bicategories of fractions · Localization · Stacks

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1 Introduction

The area of *higher geometry* deals broadly with generalisations of 'spaces', be they topological, differential geometric, algebro-geometric etc., that can be represented by groupoids (or higher groupoids) in the original category of spaces. Usually these go by the label differential, topological, algebraic etc. stacks, but when viewed as stacks there are more morphisms between objects than when viewed simply as internal groupoids; there are non-invertible maps of groupoids that become equivalences of the associated stacks. Pronk, in [8], formulated what it meant to localise a bicategory at a class of morphisms and introduced a bicategory of fractions that exists under certain conditions in order to construct this localisation. She then went on to show that 2-categories of differentiable, topological and

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algebraic stacks (of certain sorts) were indeed localisations of the 2-categories of groupoids internal to the appropriate categories.

Since then, many other cases of 2-categorical localisations have been considered, using Pronk's result applied to other categories (for extensive discussion and examples see [10, §§2,8]). However, almost all of them—only two exceptions are known to the author—deal with internal groupoids and/or stacks in some setting. In this case, the 2-category in question, and the class of morphisms at which one wants to localise, satisfy some properties making available a much simpler calculus of fractions, namely *anafunctors*. These were introduced by Makkai [7] for the category of sets *sans* Choice and in the general internal setting by Bartels [2]. The author's [10] considered the case of a sub-2-category $C \hookrightarrow \mathbf{Cat}(S)$ of the 2-category of categories internal to a subcanonical site (S, J), satisfying some mild closure conditions. The main result of [10] is that such 2-categories admit a bicategory of fractions at the so-called *weak equivalences* (also called *Morita equivalences*), and that anafunctors also calculate this localisation.

This note serves to show that given a 2-category with the structure of a *2-site* of a certain form (all covering maps must be representably fully faithful), the same result holds – namely that the bicategory of fractions of Pronk exists. One can then approach the theory of presentable stacks (on 1-sites) in a formal way, analogous to Street's formal theory of stacks [13] (cf Shulman's [12]). This result covers all others in the literature dealing with localising 2-categories of internal categories or groupoids. It may also replicate the result in [9], although the framework therein is conceptually more pleasing; the theorems of this note are definitely sufficient to imply the applications of the abstract framework of [9]

Both [9], and the recent paper [1] (written in parallel with the present note), deal with constructing localisations via fibrancy/projectivity. Hom-categories in the constructions of localisations in both papers are in fact hom-categories of the original bicategory, and so one is assured of local smallness, a problem when localising any large (bi-)category, using local smallness of the original bicategory. The present note does not assume existence of enough fibrant objects or projectives to prove local (essential) smallness (see Proposition 7). It certainly assumes less than the applications in [9] (prestacks on a subcanonical site) or [1] (internal groupoids in a regular category).

Sometimes when calculating the localisation of a 2-category of internal groupoids, various authors use what are variously known as *Hilsum-Skandalis morphisms* or *right principal bibundles* (see [10, §2] for discussion and references). In the more general setting of 2-sites as defined here such a definition is not possible, as one has hom-categories that are not groupoids. Additionally, composition of 1-arrows in the bicategory of internal groupoids and bibundles requires existence of pullback-stable reflexive coequalisers, an assumption not made here. Also, the definition of a bibundle between internal *categories* is not clear and the right notion of a map of bibundles (i.e. 2-arrows in the localisation) does not appear to be as simple as in the groupoid case.

2 Preliminaries

Though this paper touches lightly on the theory of bicategories, a knowledge of 2-categories is sufficient (an accessible reference is [6]). We consider our 2-categories to have one extra piece of structure, namely an analogue of a Grothendieck pretopology.