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Dynamical Systems and Sheaves

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Abstract

A categorical framework for modeling and analyzing systems in a broad sense is proposed. These systems should be thought of as 'machines' with inputs and outputs, carrying some sort of signal that occurs through some notion of time. Special cases include continuous and discrete dynamical systems (e.g. Moore machines). Additionally, morphisms between the different types of systems allow their translation in a common framework. A central goal is to understand the systems that result from arbitrary interconnection of component subsystems, possibly of different types, as well as establish conditions that ensure totality and determinism compositionally. The fundamental categorical tools used here include lax monoidal functors, which provide a language of compositionality, as well as sheaf theory, which flexibly captures the crucial notion of time.

Keywords Dynamical systems · Topos theory · Sheaf theory · Monoidal categories · Operads

Contents

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1 Introduction

The broad goal of this paper is to suggest a solid categorical framework for understanding and simulating systems of systems. While the current work is certainly theoretical, our ultimate interest is to make the case that category theory can be useful for modeling real-world systems, certainly not a novel thesis as we discuss in the related work section. We consider broadly defined *open* (dynamical) systems that take in, process, and send out material. Their complexity level varies greatly; for example, they can be used to model anything from an electric circuit, to a chemistry experiment, to a robot. Moreover, they are designed to be interconnected: the material output by one system is sent to, and received by, another. The central idea to which the current work adheres, building on earlier works like [31,35,39], is that a single system may arise by wiring together any number of component open subsystems; see Fig. 1. Analyzing a composite system is often intractable, because its complexity is generally exponential in the number of subsystems. Hence it is often crucial that the analysis be *compositional* [38] because such analyses can be applied to the subsystems independently, and the results can be composed in a specific sense. This also means that the analysis is robust to redesign: improvements can arise from reconfiguring any one of the numerous parts and subparts of large-scale systems (see Fig. 1) at any level of a hierarchy, which can itself be re-structured, and the analyses of unaffected systems remain valid.

The theory of monoidal categories and operads provides an excellent formalism in which to study compositionality. In our approach, an object in the symmetric monoidal category $\mathcal{W}_{\mathcal{C}}$ of $\mathcal{C}$-*labeled boxes* and *wiring diagrams* (for some category $\mathcal{C}$) looks like a box in Fig. 1, thought of as an interface for an open dynamical system with input and output ports through which it can interact with its environment. A morphism in $\mathcal{W}_{\mathcal{C}}$ describes how systems can be interconnected together to form new systems: for example, any of the two dotted composite systems in Fig. 1 are morphisms in the underlying operad, whereas that picture constitutes an operad composition of a 3-ary (top) and a 2-ary (bottom) morphism producing a 5-ary one. This framework is a cornerstone for the present work and will be described in detail; wiring diagrams become the algebraic operations for combining dynamical systems.

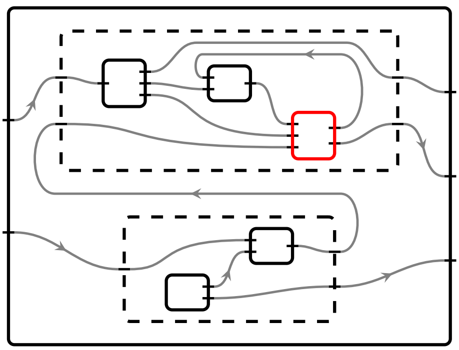

Fig. 1 Compositional analyses facilitate the rearrangement as well as the replacement of internal components