RATIONAL KRYLOV FOR NONLINEAR EIGENPROBLEMS,
AN ITERATIVE PROJECTION METHOD

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Abstract. In recent papers Ruhe suggested a rational Krylov method for nonlinear eigen-
problems knitting together a secant method for linearizing the nonlinear problem and the
Krylov method for the linearized problem. In this note we point out that the method can be
understood as an iterative projection method. Similarly to the Arnoldi method the search
space is expanded by the direction from residual inverse iteration. Numerical methods
demonstrate that the rational Krylov method can be accelerated considerably by replacing
an inner iteration by an explicit solver of projected problems.

Keywords: nonlinear eigenvalue problem, rational Krylov method, Arnoldi method, pro-
jection method

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1. Introduction

In this note we consider the nonlinear eigenvalue problem

\[(1.1) \quad A(\lambda)x = 0 \]

where \(A(\lambda) \in \mathbb{C}^{n \times n}\) is a family of matrices depending on a complex parameter \(\lambda \in D \subseteq \mathbb{C}\). As in the linear case a parameter \(\lambda\) is called an eigenvalue of problem (1.1) if the equation (1.1) has a nontrivial solution \(x \neq 0\) which is called an eigenvector corresponding to \(\lambda\). We assume in this note that the matrix \(A(\lambda)\) is large and sparse.

For sparse linear eigenproblems iterative projection methods are very efficient. Approximations to the desired eigenvalues and the corresponding eigenvectors are obtained from projections to subspaces which are expanded in the course of the algorithm. Methods of this type are the Lanczos algorithm [10], Arnoldi’s method [1] and the Jacobi-Davidson method [18], e.g., to name the most important ones. Volume [2] contains a survey and a guide to the numerical solution of eigenvalue problems.
Taking advantage of shift-and-invert techniques in Arnoldi’s method one gets approximate eigenvalues closest to the shift. Ruhe [15] generalized this approach suggesting the rational Krylov method where several shifts are used in one run. Thus one gets good approximations to all eigenvalues in a union of regions around the shifts chosen.

In some sense, Ruhe [14] generalized the rational Krylov approach to sparse nonlinear eigenvalue problems. He combined the linearization of problem (1.1) by Lagrangian interpolation and the solution of the resulting linear eigenproblem by Arnoldi’s method. Similarly to the rational Krylov process, he constructs a sequence \( V_k \) of subspaces of \( \mathbb{C}^n \). At the same time he updates Hessenberg matrices \( H_k \) which approximate the projection of \( A(\sigma)^{-1}A(\lambda_k) \) to \( V_k \). Here \( \sigma \) denotes a shift (which similarly to the rational Krylov method for linear problems can be updated in the course of the algorithm) and \( \lambda_k \) an approximation to the desired eigenvalue of (1.1). Then a Ritz vector \( x_k \) of \( H_k \) corresponding to an eigenvalue of small modulus approximates an eigenvector of the nonlinear problem from which a (hopefully) improved eigenvalue approximation of problem (1.1) is obtained.

The convergence properties of this first version of rational Krylov for nonlinear problems was far from being satisfactory. To improve its convergence, Ruhe in [16] introduced an inner iteration which enforces the residual \( r_k = A(\sigma)^{-1}A(\lambda_k)x_k \) to be orthogonal to the search space \( V_k \). This property is automatically satisfied for linear eigenproblems. The inner iteration is presented heuristically not noticing that it actually is nothing else but a solver of the projected nonlinear eigenproblem \( V_k^HA(\sigma)^{-1}A(\lambda)V_k^s = 0 \). Thus, the rational Krylov method for nonlinear eigenproblems can be interpreted as an iterative projection method, where the inner iteration can be replaced by any solver of dense nonlinear eigenproblems. Numerical examples demonstrate that the method can be accelerated considerably in this way.

Although motivated in a completely different manner the search space \( V_k \) is expanded in the same way as in the Arnoldi method for nonlinear eigenproblems introduced in [19], [20]. However, differently from rational Krylov, in the Arnoldi approach the original problem \( A(\lambda)x = 0 \) is projected to \( V_k \). Thus, the nonlinear Arnoldi method preserves symmetry properties of problem (1.1), which can be exploited when solving the projected problems.

This note is organized as follows. Section 2 summarizes the rational Krylov method as introduced by Ruhe [14], [16]. In Section 3 we give its interpretation as an iterative projection method, and we comment on modifications and improvements. Section 4 compares the original method as implemented in [7] with its modification, where the inner iteration is replaced by a direct solver of the projected problem, and with the Arnoldi method for a rational eigenproblem governing mechanical vibrations of a fluid-solid structure.