Abstract. We give an explicit expression of a two-parameter family of Flensted-Jensen’s functions $\Psi_{\mu,\alpha}$ on a concrete realization of the universal covering group of $U(1,1)$. We prove that these functions are, up to a phase factor, radial eigenfunctions of the Landau Hamiltonian on the hyperbolic disc with a magnetic field strength proportional to $\mu$, and corresponding to the eigenvalue $4\alpha(\alpha - 1)$.

Keywords: Flensted-Jensen’s functions, universal covering group, Landau Hamiltonian, hyperbolic disc

MSC 2000: 33C05, 35J10, 35Q40, 43A90, 57M10, 58C40

1. Introduction

In quantum mechanics, considerable attention has been paid to the physics of a charged particle evolving in a plane under the influence of a perpendicular uniform magnetic field. This problem, called the planar Landau problem [1], has been generalized to two-dimensional curved surfaces with a normal stationary magnetic field [2].

For a charged particle evolving in the hyperbolic plane under the influence of a uniform magnetic field, eigenfunctions and eigenvalues of the corresponding Landau Hamiltonian have been discussed in the context of the spectral theory [3]. Eigenstates can also be obtained as representation coefficients of the Lie group describing the symmetry of the quantum system [4]. Spherical functions are special coefficients of group representations, which are usually indexed by a spectral parameter.

Here our main aim is to attach to the Landau problem on the hyperbolic disc a set of Flensted-Jensen’s (FJ’s) spherical functions [5] defined on the universal covering group of $U(1,1)$. We prove that these FJ’s functions are eigenstates of the particle.
The advantage of these functions is that they are indexed by two parameters, one of them being proportional to the magnetic field strength while the other occurs in the parametrization of the eigenvalue of the Hamiltonian.

The paper is organized as follows. In Section 2 we discuss the hyperbolic Landau Hamiltonian. In Section 3, we give the general form of eigenfunctions of the Hamiltonian. Section 4 deals with a realization of the universal covering group of $U(1, 1)$. An Iwasawa decomposition corresponding to this realization is given in Section 5. In Section 6, we give an explicit expression of a two-parameter family of FJ’s spherical functions and we show that it constitutes a family of eigenstates of the particle. Section 7 is devoted to some concluding remarks.

2. The Landau Hamiltonian on the unit disc

Let $H^2_0 := \{ \xi \in \mathbb{C}, \Im \xi > 0 \}$ be the upper halfplane endowed with the metric $\text{d}s^2 = a^2 y^{-2} (\text{d}x^2 + \text{d}y^2)$, where $x = \Re \xi$, $y = \Im \xi$, and $a > 0$ is the parameter related to the curvature $\kappa$ by $\kappa = -2/a^2$. The area element $\text{d}\mu_a(\xi)$ has the form $\text{d}\mu_a(\xi) = a^2 y^{-2} \text{d}x \wedge \text{d}y$. A constant uniform magnetic field on $H^2_0$ is given by a 2-form $\mathbf{B}$ defined as

$$\mathbf{B} = \frac{Ba^2}{y^2} \text{d}x \wedge \text{d}y,$$

where $B > 0$ is the field intensity. The form $\mathbf{B}$ can be represented as $\mathbf{B} = \mathbf{d}\mathbf{A}$, where $\mathbf{A} = Ba^2 y^{-1} \text{d}x$ is the vector potential (Landau gauge) we have chosen.

The Schrödinger operator describing a particle of charge $e$ and mass $m_*$ which lives on $H^2_0$ and interacts with the magnetic field $\mathbf{B}$ is given by

$$H_b := -\frac{\hbar^2}{2m_*a^2} (y^2 (\partial_x^2 + \partial_y^2) - 2iby\partial_x - b^2)$$

where $b$ is the dimensionless quantity $b := eBa^2/\hbar c$. For the sake of simplicity, we put $a = e = c = \hbar = 2m_* = 1$. We denote $\mathbf{H}^2 := H^2_0$ and will consider a slight modification of $H_b$ in (2.1). Actually, we will deal with the operator ([6], p. 8073):

$$H_B := y^2 (\partial_x^2 + \partial_y^2) - 2iBy\partial_x$$

with $C_0^\infty (\mathbf{H}^2)$ as its regular domain in the Hilbert space $\mathcal{H} := L^2(\mathbf{H}^2, y^{-2} \text{d}x \text{d}y)$. The spectrum of $H_B$ in $\mathcal{H}$ consists of two parts: (i) an absolutely continuous spectrum in the interval $(-\infty, 0]$, (ii) a point spectrum consisting of a finite number of infinitely degenerate eigenvalues of the form ([7], p. 11)

$$E_m^B := (B - m)(B - m - 1), \quad 0 \leq m < B - \frac{1}{2}$$

when $2B > 1$. 98