RELATION BETWEEN ALGEBRAIC AND GEOMETRIC VIEW ON NURBS TENSOR PRODUCT SURFACES*

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Abstract. NURBS (Non-Uniform Rational B-Splines) belong to special approximation curves and surfaces which are described by control points with weights and B-spline basis functions. They are often used in modern areas of computer graphics as free-form modelling, modelling of processes. In literature, NURBS surfaces are often called tensor product surfaces. In this article we try to explain the relationship between the classic algebraic point of view and the practical geometrical application on NURBS.

Keywords: tensor product surface, bilinear form, B-spline, NURBS

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1. Introduction

NURBS have become a standard type of mathematical approximation of surfaces in modern computer graphics. The general NURBS surface is described by a net of control points with weights and by two knot vectors. Theory of NURBS is summarized in [9], for example. NURBS objects are often used for free-form modelling because of their good modification possibilities (e.g. technic FFD—see [12]). The NURBS are used in different branches, for example robotics [5], film industry [13], reverse engineering [10], GIS [14], physical computing [11], etc.

The basic B-spline theory was proposed by Carl de Boor in [1], where the tensor product is schematically described. Tensor calculus is described in [2] and [6]. Non-tensor product NURBS surfaces using the smoothing cofactor-conformality method are constructed in [7].

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In this paper, we try to explain the algebraic point of view on tensor product surfaces and to establish the connection between this theory and the geometrical and practical use of NURBS.

Section 2 briefly outlines the basic tensor theory. In Section 3 the algebraic approach to B-spline functions and curves is discussed. Section 4 discusses the projective extension of NURBS curves and defines the abstract curve as a set of curves which are invariant with each other.

In Section 5 we deal with the NURBS surfaces. Analogously to Section 4, we introduce an abstract surface based on the characteristic form. In the last Section 6, we discuss some results and practical examples of our theory.

2. Tensor calculus

Let $U$, $V$ be vector spaces over a field $\mathbb{T}$. A bilinear form $\omega$ on $U \times V$ is a function $\omega: U \times V \to T$ which satisfies the well-known axioms. The vector space of all bilinear forms between spaces $U$ and $V$ is called the tensor product. These mappings can be written as

\begin{equation}
\omega(u, v) = \sum_{i=0}^{m} \sum_{j=0}^{n} u_i v_j \omega(e_i, f_j) = \sum_{i=0}^{m} \sum_{j=0}^{n} u_i v_j a_{ij}, \quad a_{ij} = (e_i, f_j),
\end{equation}

or in the matrix form

\begin{equation}
\omega(u, v) = (u_0, u_1, \ldots, u_m) \begin{pmatrix} a_{00} & a_{01} & \cdots & a_{0n} \\
 a_{10} & a_{11} & \cdots & a_{1n} \\
 \vdots  & \vdots  & \ddots & \vdots  \\
 a_{m0} & a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} v_0 \\
 v_1 \\
 \vdots \\
 v_n \end{pmatrix},
\end{equation}

where $u_i, i = 0, \ldots, m, v_j, j = 0, \ldots, n$ are the coordinates of the vectors $u \in U, v \in V$ with bases $\{e_0, e_1, \ldots, e_m\}, \{f_0, f_1, \ldots, f_n\}$.

A surface is described by a function of two parameters as a mapping of a plane area to Euclidean 3-dimensional space. Formally

\begin{equation}
S(s, t) = (x(s, t), y(s, t), z(s, t)) = \sum_{i=0}^{m} \sum_{j=0}^{n} f_i(u)g_j(v)b_{ij}
\end{equation}

where $b_{ij} = (x_{ij}, y_{ij}, z_{ij}), 0 \leq s, t \leq 1, m = 2, n = 2$.

Eq. (2.1) is formally similar to Eq. (2.3). Therefore, NURBS surface defined by Eq. (2.3) is often called as the tensor product surface (see [9]).