ON THE GENERALIZATIONS OF KALANDIA’S LEMMA

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Abstract

Based on Bernstein’s Theorem, Kalandia’s Lemma describes the error estimate and the smoothness of the remainder under the second part of Hölder norm when a Hölder function is approximated by its best polynomial approximation. In this paper, Kalandia’s Lemma is generalized to the cases that the best polynomial is replaced by one of its four kinds of Chebyshev polynomial expansions, the error estimates of the remainder are given out under Hölder norm or the weighted Hölder norms.

Key words Chebyshev polynomials, Chebyshev norm, weighted Hölder norm

AMS(2000) subject classification 41A10, 41A58

1 Introduction

Let \( f(x) \in H^{\mu}[-1, 1] \) be a Hölder continuous function with \( 0 < \mu < 1 \), and its Hölder norm \( \|f\|_{H^{\mu}[-1, 1]} \) is defined as

\[
\|f\|_{H^{\mu}[-1, 1]} = \|f\|_\infty + M[f, \mu],
\]

where \( \|f\|_\infty = \max_{-1 \leq x \leq 1} |f(x)| \) is called Chebyshev norm of \( f \), and

\[
M[f, \mu] = \sup_{x_1, x_2 \in [-1, 1]} \frac{|f(x_1) - f(x_2)|}{|x_1 - x_2|^\mu}
\]

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is called the Hölder constant of $f$.

In approximation theory Jackson Theorem\textsuperscript{[3]} tells us that the best error estimate of the remainder under Chebyshev norm is $O(n^{-\mu})$ when $f(x) \in H^\mu[-1, 1]$ is approximated by its best polynomial approximation $p_n(x)$. In details,

$$E_n(f) \leq O(1) \omega \left( f, \frac{1}{n} \right) \leq O(1) M[f, \mu] n^{-\mu} \leq O(1) \| f \|_{H^\mu[-1, 1]} n^{-\mu},$$  \hspace{1cm} (1.3)

where $E_n(f) = \max_{-1 \leq x \leq 1} |f(x) - p_n(x)|$, $\omega \left( f, \frac{1}{n} \right) = \max_{\frac{1}{n} \leq |x_1 - x_2| \leq \frac{1}{n}} |f(x_1) - f(x_2)|$ is the modulus of continuity of $f$. Obviously, $\omega \left( f, \frac{1}{n} \right) \leq M[f, \mu] \max_{|x_1 - x_2| \leq \frac{1}{n}} |x_1 - x_2|^\mu \leq \| f \|_{H^\mu[-1, 1]} n^{-\mu}$. For simplicity, sometimes we will use $O(1)$ to represent a constant independent of $f$ and $n$.

A. I. Kalandia described in [4] the smoothness of $f(x) - p_n(x)$ under the second part of Hölder norm, that is,

**Kalandia’s Lemma\textsuperscript{[4]}**. Let $f(x) \in H^\mu[-1, 1]$, if there is a polynomial $p_n(x)$ such that $|f(x) - p_n(x)| \leq C_1 n^{-\mu}$, then we have

$$M[f, \nu] \leq C_2 n^{2\nu - \mu}, \quad 0 < 2\nu < \mu,$$  \hspace{1cm} (1.4)

where $f_n(x) = f(x) - p_n(x)$, $C_1, C_2$ are two constants independent of $x$ and $n$.

We can know from Bernstein’s Theorem and Kalandia’s Lemma that the best error estimate of $f(x) - p_n(x)$ under Hölder norm is

$$\| f - p_n \|_{H^\mu[-1, 1]} \leq O(1) \| f \|_{H^\mu[-1, 1]} n^{2\nu - \mu}, \quad 0 < 2\nu < \mu.$$  \hspace{1cm} (1.5)

The results (1.3), (1.4) and (1.5) are very useful for numerical method of various singular integral equations\textsuperscript{[1,2,4]}. In fact, Kalandia’s Lemma came from the numerical method of a singular integral equation in the theory of wing\textsuperscript{[4]}. It can be used in the numerical method of Singular Integral Equation of the first kind, because $p_n(x)$ in (1.4) will be replaced by $n$-th partial sums of Chebyshev polynomial series of $f$. The approximating errors under Chebyshev norm or weighted Chebyshev norm have been solved in [3, 5], but the error estimates under the second part of Hölder norm are not discussed yet, hence it is worth while to study the generalization of Kalandia’s Lemma, which is the purpose of this paper.

In section 2, some well-known results on Chebyshev polynomials are introduced for later use. In section 3, Kalandia’s Lemma is generalized to the case of first kinds of Chebyshev weight, the main idea comes from [4], but calculations are more complicated and delicate. This result can be found in [2] too. In section 4, the weighted Hölder norms are introduced, so that Kalandia’s Lemma could be generalized to the other three cases of Chebyshev weights.