Stability of Information Equilibrium in Reflexive Games

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Abstract—For the reflexive game where the agents’ decisions rely on a hierarchy of beliefs about essential parameters, beliefs about beliefs, and so on, consideration was given to stability of its information equilibrium. Stability lies in that the expected result of the game is observed precisely by each participant, be it real and phantom, that is, existing in the belief of other real or phantom participants.

1. INTRODUCTION

Reflexive games based on the concept of information equilibrium represent one of the means of game-theoretical modeling of decision making under incomplete information [1–3]. We recall that the reflexive game is defined by the cortege \( \{N, (X_i)_{i \in N}, f_i(\cdot)_{i \in N}, \Omega, I\} \), where \( N = \{1, 2, \ldots, n\} \) is the set of game participants (players, agents), \( X_i \) is the set of feasible actions \( x_i \) of the \( i \)th agent, \( f_i(\cdot) : \Omega \times X_i \rightarrow 2^{\Omega} \) is its goal function, \( i \in N \), \( X' = \prod_{i \in N} X_i \), \( \theta_i \in \Omega \) is its belief about the state of nature \( \theta \in \Omega \), \( \theta_{ij} \in \Omega \) is its belief about the beliefs of the \( j \)th agent, \( \theta_{ijk} \in \Omega \) are the beliefs of \( \ell \)th agent about the ideas of the \( j \)th agent about the beliefs of the \( k \)th agent, and so on, in the general case, indefinitely. The entire collection of mutual beliefs makes up the information structure \( I = (I_1, I_2, \ldots, I_n) \), where \( I_i = (\theta_i, \theta_{ij}, \theta_{ijk}, \ldots) \), \( i, j, k \in N \), is the information structure of the \( i \)th agent [1–3]. The subscripts \( i, j, k, \ldots \) run all values from \( N \) in such a way that the information structure \( I \) is an \( n \)-ary tree whose vertices are the elements of the set \( \Omega (\theta_i, \theta_{ij}, \theta_{ijk}, \text{and so on}) \) and the edges connect \( \theta_i \) and \( \theta_{ij} \), \( \theta_{ij} \) and \( \theta_{ijk} \), and so on.

We denote by \( \Sigma_{+} \) the set of every possible finite sequences of the subscripts from \( N \) and by \( \Sigma \) the union of \( \Sigma_{+} \) and an empty sequence. Let \( i \in N \), \( \tau \in \Sigma \). Then, we denote \( I_{\tau i} = (\theta_{\tau i}, \theta_{\tau ij}, \theta_{\tau ijk}, \ldots) \), \( i, j, k \in N \).

Definition. The set of actions \( x^*_{\tau i}, \tau \in \Sigma, x^*_{\tau i} \in X_i \), is called the information equilibrium if the following conditions are met:

1. the information structure \( I \) has a finite complexity \( \nu \), that is, the tree \( I \) comprises a finite number \( \nu \) of pairwise distinct subtrees;
2. \( \forall i \in N, \forall \lambda, \mu \in \Sigma \), \( I_{\lambda i} = I_{\mu i} \Rightarrow x^*_{\lambda i} = x^*_{\mu i} \);
3. \( \forall i \in N, \forall \sigma \in \Sigma \), \( x^*_{\sigma i} \in \text{Arg} \max_{x_i \in X_i} f_i(\theta_{\sigma i}, x^*_{\sigma i1}, \ldots, x^*_{\sigma i,j-1}, x_i, x^*_{\sigma i,j+1}, x^*_{\sigma i,n}) \).

Information equilibrium is a generalization of the most popular concept of solving the noncooperative game that is known as the Nash equilibrium [4, 5]. It passes into the Nash equilibrium if the state of nature \( \theta \) is the common knowledge [5, 6] of the agents. Existence conditions for information equilibrium that are similar to those of the Nash equilibrium can be found in [1, 2]. The reader can find in [1, 7] many examples of using the concept of information equilibrium in the...
socio-economic applications. We note that the Bayes games introduced by J. Harsanyi [8] offer an alternative to modeling the situations with incomplete information. The “reflexive” and “Bayes” approaches are compared in [9].

2. STABLE INFORMATION EQUILIBRIUM

One of the main distinctions of the “classical” Nash equilibrium lies in its self-sustaining nature: if a game is repeated more than once and all players but the \(i\)th one take the same equilibrium actions, then the \(i\)th player has no reason to deviate from its equilibrium action. This is related, obviously, with the fact that the beliefs of all players about reality are adequate.

In the case of information equilibrium, the situation, generally speaking, may differ. Indeed, it may happen as the consequence of a single play of the game that some (or even all) players observe a result other than the expected one. This may be due both to an incorrect belief about the state of nature and inadequate information about the beliefs of the opponents. Self-sustained nature of equilibrium is violated anyhow: if a game is repeated, the player actions can vary.

In some cases, however, self-sustained equilibrium can take place for different—and, generally speaking incorrect—beliefs of the agents. Informally, this happens if each agent—both real and phantom, that is, existing in mind of other real or phantom agents, [1–3]—observes the expected result. Formalization needs an extended description of the reflexive game.

We complement the definition of the reflexive game (see above) by a collection of functions \(w_i(\cdot): \Omega \times X' \rightarrow W_i, i \in N\), mapping each \((\theta, x)\), where \(x = (x_1, x_2, \ldots, x_n) \in X'\), into the element \(w_i\) of some set \(W_i\). This element \(w_i\) is what the \(i\)th agent observes as the result of playing the game.

The function \(w_i(\cdot)\) will be called the observation function of the \(i\)th agent. We assume that the functional dependences \(w_i(\cdot)\) are common knowledge [1, 6] of the agents.

If \(w_i(\theta, x) = (\theta, x)\), that is, \(W_i = \Omega \times X'\), then the \(i\)th agent observes both the state of nature and the actions of all agents. If, on the contrary, the set \(W_i\) consists of one element, then the \(i\)th agent observes nothing.

Let there be information equilibrium \(x_\tau, \tau \in \Sigma_+\), in the reflexive game. We fix \(i \in N\) and consider the \(i\)th agent that expects to observe the value

\[w_i(\theta_1, x_1; \ldots, x_{i-1}, x_i, x_{i+1}; \ldots, x_n)\].

(1)

As a matter of fact, it observes

\[w_i(\theta, x_1; \ldots, x_{i-1}, x_i, x_{i+1}; \ldots, x_n)\].

(2)

Therefore, the requirement of stability for the \(i\)th agent implies that (1) and (2), which are the elements of some set \(W_i\), coincide.

Let (1) and (2) be equal, that is, after playing game the \(i\)th agent does not suspect that its beliefs are not true. Is this fact nevertheless sufficient for taking the same action \(x_i\) to replay the game? Clearly, the answer is negative as is illustrated by the following example.

**Example 1.** Let in the reflexive bimatrix game where \(\Omega = \{1, 2\}\) the gains be defined by the bimatrices (agent 1 chooses row and agent 2 column, that is, \(X_1 = X_2 = \{1; 2\}\)) and at that the second agent regard \(\theta = 2\) as common knowledge and the first agent know the real state of nature \(\theta = 1\) and be adequately informed about the second agent. Stated differently, \(\theta = \theta_1 = 1, \theta_2 = \theta_{12} = 2\) for any sequence of subscripts \(\sigma \in \Sigma\)

\[
\begin{pmatrix}
1, 1 & 0, 0 \\
0, 1 & 2, 0
\end{pmatrix}
\begin{pmatrix}
0, 1 & 1, 2 \\
1, 1 & 2, 2
\end{pmatrix}
\]

Let also each agent observe its gain and this be the common knowledge.