MULTI-IMPLIED PEER TWO-STEP W-METHODS FOR PARALLEL TIME INTEGRATION

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Abstract.

Peer two-step W-methods are designed for integration of stiff initial value problems with parallelism across the method. The essential feature is that in each time step a ‘peer’ approximations are employed having similar properties. In fact, no primary solution variable is distinguished. Parallel implementation of these stages is easy since information from one previous time step is used only and the different linear systems may be solved simultaneously. This paper introduces a subclass having order $s - 1$ where optimal damping for stiff problems is obtained by using different system parameters in different stages. Favourable properties of this subclass are uniform stability for realistic stepsize sequences and a superconvergence property which is proved using a polynomial collocation formulation. Numerical tests on a shared memory computer of a matrix-free implementation with Krylov methods are included.

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1 Introduction.

Recently the authors introduced a new class of time integration methods for the solution of large and stiff initial value problems

\begin{equation}
y' = f(t, y), \quad t_0 \leq t \leq t_e, \quad y(t_0) = y_0 \in \mathbb{R}^n,
\end{equation}

[10]. These methods have an inherent parallelism across the method since they employ $s$ stages which are totally independent within the actual time step. These stages depend on values from the previous step only leading to a two-step structure. An important difference to most standard schemes is that no extraordinary solution variable for $y$ with distinguished properties is computed.
All $s$ stage solutions are peers sharing essentially the same accuracy and stability properties. As an example, singly-implicit methods being almost L-stable exist up to order 7 [10]. Also, the methods do not suffer from order reduction in stiff problems [11]. Method parallelism may be attractive in a computing environment with moderate parallelism using a black-box implementation of the right-hand side $f$. On large scale computers this parallelism still may simplify load balancing in adaptive space discretizations. Many approaches for parallel integration methods use parallel iteration schemes for non-parallel methods, e.g., [2, 6, 1]. A recent survey on classes of General Linear Methods with inherent parallelism can be found in [3].

The form of the ‘Parallel Peer two-Step W-Methods’ (PPSW-methods) is as follows. In each time step from $t_m$ to $t_{m+1} = t_m + h_m$ solutions $Y_{mi} \cong y(t_{mi})$, $i = 1, \ldots, s$, are computed as approximations at the points

$$t_{mi} := t_m + h_{mi}, \quad i = 1, \ldots, s,$$

which are not restricted to the time interval $[t_m, t_{m+1}]$. This means that the off-step nodes $c_i$ are not confined to $[0, 1]$. The time step consists of $s$ linearly implicit stages, $i = 1, \ldots, s$,

$$\begin{align*}
(I - \gamma_i h_m T_m) Y_{mi} &= \sum_{j=1}^{s} (b_{ij} I + h_m \gamma_{ij} T_m) Y_{m-1,j} + \\
&+ h_m \sum_{j=1}^{s} a_{ij} f(t_{m-1,j}, Y_{m-1,j}),
\end{align*}$$

where the reals $\gamma_i > 0$, $b_{ij}, \gamma_{ij}, a_{ij}$ are the parameters of the method. The matrix $T_m$ should be an approximation of the Jacobian $J_f(t_m, y(t_m))$ for stability reasons only. We point out that the right-hand side in (1.3) only uses information $Y_{m-1,j}$ from the previous time step and the left-hand side describes $s$ independent linear systems. The solution of these systems is expected to cause the main computational effort for realistic stiff problems of large dimension $n$. In [10] the first two authors analyzed the simpler, singly-implicit case for these schemes where the matrix in all $s$ stage equations is the same, $\gamma_i \equiv \gamma$, $i = 1, \ldots, s$. Such schemes may be used in sequential computations and it was found in [10], indeed, that they may already be competitive in this setting with existing standard codes. Still, the results were not fully satisfactory since singly-implicit methods lack optimal damping properties for very stiff problems. For parallel implementations the general multi-implicit case with different $\gamma_i$ is appropriate. With these methods optimal stiff damping is possible with the choice

$$A = (a_{ij})_{i,j=1}^{s} = -\Gamma, \quad \Gamma = (\gamma_{ij})_{i,j=1}^{s}$$

and we will consider only such methods in this paper. The diagonal matrix of the system parameters $\gamma_i$ is denoted by $G := \text{diag}(\gamma_i)$.

While stability for very stiff problems is easily accomplished for the schemes (1.3), zero stability is not. This problem is aggravated through the fact that the coefficient matrices of nontrivial methods depend on the stepsize ratio

$$\sigma_m := h_m / h_{m-1} \leq \bar{\sigma},$$