EFFICIENT REALIZATION OF THE GALERKIN METHOD IN VIEW OF NEW PROPERTIES OF CHEBYSHEV POLYNOMIALS

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The Galerkin method is used taking account of new properties of Chebyshev polynomials of the first and second kinds to demonstrate the possibility of simplifying the asymptotic analysis and synthesis of various systems governed by operator equations.

\textbf{Keywords:} Galerkin method, Hilbert space, scalar product, Chebyshev polynomials.

INTRODUCTION

Analysis of a wide class of estimation, filtration, identification, and optimal control problems, etc. can be reduced to operator equations in given functional spaces \([1–6]\). Approximate solution of such equations allows us to analyze and synthesize various dynamic systems, both uncontrolled \([5]\) and controlled \([6]\).

Let \(F(z)\) be a nonlinear operator acting from a Hilbert space \(X\) into a Hilbert space \(Y\):

\[
F(z) = h, \quad z = z(x), \quad h = h(x), \quad z \in X, \quad h \in Y. \tag{1}
\]

The Galerkin method

\[
\left\langle F \left( \sum_{n=0}^{N} q_n \varphi_n(x) \right), \kappa_j(x) \rightangle = \langle h(x), \kappa_j(x) \rangle, \quad j = 0, 1, \ldots, N, \tag{2}
\]

is frequently used for approximate solution of Eq. (1). Here \(\{\varphi_n(x)\}_{n=0}^{N}\) is the coordinate system of functions, \(\{\kappa_j(x)\}_{j=0}^{N}\) is the projective system of functions, and \(\{q_n\}_{n=0}^{N}\) are unknown numerical coefficients.

Solving the system of equations (2), we find the family of desired coefficients \(\{q_n\}_{n=0}^{N}\), and then construct the approximate solution of the operator equation (1) as the generalized polynomial

\[
\tilde{z}(x) = \sum_{n=0}^{N} q_n \varphi_n(x). \tag{3}
\]

The general formulation of the problem does not guarantee that system (2) has at least one solution. If (2) has a unique solution for each \(N = 0, 1, 2, \ldots\), then the approximate solution \(\tilde{z}(x)\) may not converge even weakly to the exact solution of Eq. (1) as \(N \to \infty\). Nevertheless, the Galerkin method is a powerful tool not only for deriving approximate solutions but also for proving the existence theorems for the solutions of linear and nonlinear operator equations.

We will use Chebyshev polynomials of the first and second kinds as coordinate and projective functions. It will be shown that the realization of the Galerkin method (1)–(3) can be considerably simplified for a wide class of operator equations.
FORMULATION OF THE PROBLEM

Consider Chebyshev polynomials $G_{ni} = G_{ni}(x)$ of the first and second kinds and the following recurrent formula (for $x \in (-\infty, \infty)$):

$$G_{ni}(x) = 2xG_{n-1,i}(x) - G_{n-2,i}(x), \quad i = 1, 2, n = 2, 3, \ldots,$$

where $G_{n1}(x) = T_n(x)$ and $G_{n2}(x) = U_n(x)$ are the Chebyshev polynomials of the first and second kinds, respectively (the notation $T_n(x)$ and $U_n(x)$ is borrowed from [6]).

We complement this formula with the initial conditions

$$G_{01}(x) = T_0(x) = T_0 = 1, \quad G_{11}(x) = T_1(x) = T_1 = x,$$

$$G_{02}(x) = U_0(x) = U_0 = 1, \quad G_{12}(x) = U_1(x) = U_1 = 2x.$$

Since it is common practice to define functions on bounded domains, we will use the following representations:

$$G_{n1}(x) = \cos(n \arccos(x)), \quad |x| \leq 1, \ n = 0, \pm 1, \ldots,$$

$$G_{n2}(x) = (1-x^2)^{-1/2} \sin((n+1)\arccos(x)), \quad |x| < 1, \ n = 0, \pm 1, \ldots.$$

In practical realization of various numerical methods based on Chebyshev polynomials, the need frequently arises to calculate the following scalar products [3, 4]:

$$\langle x^d G_{ni}^{(m)} , x^u G_{pi}^{(r)} \rangle = \int_{-1}^{1} x^d + u G_{ni}^{(m)} G_{pi}^{(r)} h_i(x) \, dx, \quad i = 1, 2, \ m \leq n, \ r \leq p, \quad (4)$$

where $h_1(x) = (1-x^2)^{-1/2}$, $h_2(x) = (1-x^2)^{1/2}$, $d, u, n, m, r \in 0, 1, \ldots$, $G_{ni}^{(m)} = (d^m / dx^m) G_{ni}$, $G_{pi}^{(r)} = (d^r / dx^r) G_{pi}$.

Since the use of the integral relation (4) for repeated calculations of the scalar products $\langle x^d G_{ni}^{(m)} , x^u G_{pi}^{(r)} \rangle$ involves much computational effort, we will attempt to derive finite analytical expressions and to establish the regularity of zeroing $\langle x^d G_{ni}^{(m)} , x^u G_{pi}^{(r)} \rangle$ depending on the values of the indices $d, u, n, p, m, r$.

PROPERTIES OF CHEBYSHEV POLYNOMIALS

The explicit expressions for scalar products (4) follow from the following theorem.

**THEOREM.** If $n, p \geq 1$ and $d, u, m, r \geq 0$, then the scalar products $\langle x^d G_{ni}^{(m)} , x^u G_{pi}^{(r)} \rangle$ of type (4) are as follows: for $p + n - (m + r) + d + u = 2q$, $q \in 0, 1, \ldots$,

$$\langle x^d G_{ni}^{(m)} , x^u G_{pi}^{(r)} \rangle = \pi \sum_{j = 0}^{[(p+1)/2]} \sum_{k = 0}^{[(n-m)/2]} C_{pqj}^i C_{nmk}^i \frac{(q_{jk} - 1)!!}{(q_{jk} + 2(i - 1))!!}, \quad i = 1, 2, \quad (5)$$

for $p + n - (m + r) + d + u = 2q + 1$, $q \in 0, 1, \ldots$,

$$\langle x^d G_{ni}^{(m)} , x^u G_{pi}^{(r)} \rangle = 0, \quad i = 1, 2, \quad (6)$$

where $\lfloor \cdot \rfloor$ denotes the integer part of a number.