INVARIANCE THEOREMS FOR A CLASS OF SYSTEMS OF RANDOM NONLINEAR EQUATIONS OVER AN ARBITRARY FINITE RING WITH LEFT UNITY

A. A. Levitskaya

A class of homogenous systems of random nonlinear equations over an arbitrary finite ring with left unity is considered. The author analyzes the invariance boundaries for limit factorial moments of nonzero solutions, the limit distribution of the number of nonzero solutions, and the geometrical structure of the set of nonzero solutions of the system as the number of unknowns tends to infinity.

Keywords: system of random nonlinear (linear) equations over a finite ring with left unity (over a finite field), factorial moment of the number of nonzero solutions of a system, distribution of the number of nonzero solutions of a system.

Let $R$ be an arbitrary finite ring with left unity. Assume that $R \neq \text{GF}(2)$. Consider the system of equations

$$
\sum_{k=1}^{d_i} \sum_{1 \leq j_1 \leq j_2 \leq \ldots \leq j_k \leq n} a_{i,j_1\ldots j_k} x_{j_1} \cdots x_{j_k} = 0, \quad i = 1, n-s,
$$

over $R$ and additionally, if $R$ is not commutative, consider one more system

$$
\sum_{k=1}^{d_i} \sum_{1 \leq j_1, j_2, \ldots, j_k \leq n} a_{i,j_1\ldots j_k} x_{j_1} \cdots x_{j_k} = 0, \quad i = 1, n-s,
$$

where $s$ is an integer constant with arbitrary sign, $a_{i,j_1\ldots j_k}$ are random variables independent in aggregate, and $d_i$, $2 \leq d_i \leq n$, $i = 1, n-s$, are natural numbers.

To save space, we will analyze systems (1) and $(\Gamma)$ simultaneously. If there are differences between the concepts and theoretical conclusions concerning (1) and $(\Gamma)$, we will comment them.

Let $\nu_n$ be the number of solutions of (1) ($(\Gamma)$) that differ from the zero vector $0 = (0 \ldots 0)$.

We will analyze systems (1), $(\Gamma)$ provided that the distributions of the random variables $a_{i,j_1\ldots j_k}$ satisfy the constraints

$$
\frac{l_0}{m(l_0-1)} \cdot \ln c_n n = \delta_n \leq P(a_{i,j_1\ldots j_k} = z), \quad z \in R,
$$

$$
1 \leq j_1, \ldots, j_k \leq n, \quad k = \frac{1}{d_i}, \quad i = 1, n-s,
$$

where $l_0$ equals the least cardinality of the nonzero left ideals of $R$, $m$ is the cardinality of the ring $R$, and $c_n$ is an arbitrary sequence tending to $\infty$ as $n \to \infty$. The purpose of our study is as follows: given characteristics of system (1) ($(\Gamma)$) as $n \to \infty$, find the limit factorial moments $\nu_n$ and use the results to study the limit distribution of $\nu_n$, and to describe the geometrical structure of the set of solutions of (1) ($(\Gamma)$).
A similar problem is solved in \cite{1} for system (1) over \( \mathbf{R} = \mathbf{GF}(q) \), where the range of summation under the sign of the second sum in (1) has the form \( 1 \leq j_1 < j_2 < \ldots < j_k \leq n \), \( k = \overline{1,d_j}, \ i = \overline{1,n-s} \).

As in \cite{1}, along with systems (1), (T), let us introduce a corresponding linear system

\[
\sum_{j=1}^{n} a_{i,j} \cdot x_j = 0, \ i = \overline{1,n-s},
\]

where \( a_{i,j}, \ i = \overline{1,n-s}, \ j = \overline{1,n}, \) are random variables, independent in aggregate, whose distributions satisfy constraints similar to (2), i.e.,

\[
\frac{l_0}{m(l_0-1)} \cdot \ln \frac{c_n \cdot n}{n} = \delta_n \leq p(a_{i,j} = z), \ z \in \mathbf{R}, \ i = \overline{1,n-s}, \ j = \overline{1,n}.
\]

Below, we will use the following notation: \( v_{0n} \) is the number of nonzero solutions of (3); \( R' = \mathbf{R} \times \ldots \times \mathbf{R} \) is a vector space over \( \mathbf{R} \) whose elements are \( n \)-dimensional column vectors; \( u = \begin{pmatrix} u_1 \\ \vdots \\ u_r \end{pmatrix} \) (superscripts are possible) is an element of \( R' \); \( 0 \) is the unity of the additive group \( R' \); \( I \) (subscripts are possible) is a left ideal of \( \mathbf{R} \); \( I' = \begin{pmatrix} \begin{pmatrix} z_1 \\ \vdots \\ z_r \end{pmatrix} \end{pmatrix}, z_i \in I, i = \overline{1,r} \) is an \( n \)-dimensional vector space over \( I; I_1 \times I_2 \times \ldots \times I_r = \begin{pmatrix} \begin{pmatrix} z_1 \\ \vdots \\ z_r \end{pmatrix} \end{pmatrix}, z_i \in I_i, i = \overline{1,r}; I \cdot I = \{ z_1 \cdot z_2: z_1, z_2 \in I \}; M_r \) (superscripts are possible) is a left submodule of \( R' \); \( u^1 \cdot u^2 = \begin{pmatrix} u^1_1 u^2_1 \\ \vdots \\ u^1_r u^2_r \end{pmatrix}, u^1, u^2 \in R' \); \( z u^2 = \begin{pmatrix} z u^1_1 \\ \vdots \\ z u^1_r \end{pmatrix}, u \in R' \); \( (M_r)^2 = \{ u \cdot v: u, v \in M_r \}; x = (x_1, \ldots, x_n) \) (superscripts are possible) is an \( n \)-dimensional vector whose coordinates are elements of \( \mathbf{R}; X_r = \begin{pmatrix} x^1 \\ \vdots \\ x^r \end{pmatrix} \); \( a_j X_r = \sum_{k=1}^{d_j} a_{i,j-1,j_k} \begin{pmatrix} x^1_j \\ \vdots \\ x^r_j \end{pmatrix}, a_j \in \mathbf{R}, \ n_j \in \mathbf{Z}^+, \ j = \overline{1,n} \); \( i = \overline{1,n-s}, \) where the summation range is \( \mathbf{J} = \{ 1 \leq j_1 \leq \ldots \leq j_k \leq n \} \) in case of system (1) and \( \mathbf{J} = \{ 1 \leq j_1, \ldots, j_k \leq n \} \) for system (T);

\[
a_{i,j}^0 X_r = \sum_{j=1}^{n} a_{i,j} \begin{pmatrix} x^1_j \\ \vdots \\ x^r_j \end{pmatrix} = \overline{1,n-s}, \quad \mathbf{N}_{M_r} = \{ X_r: \sum_{j=1}^{n} a_{i,j} \begin{pmatrix} x^1_j \\ \vdots \\ x^r_j \end{pmatrix} + \sum_{j=1}^{n} \alpha_j \begin{pmatrix} x^1_j \\ \vdots \\ x^r_j \end{pmatrix}, a_{i,j} \in \mathbf{R}, \ n_j \in \mathbf{Z}^+, \ j = \overline{1,n} \} = M_r, \ i.e.,
\]

\( \mathbf{N}_{M_r} \) — is the set of matrices \( X_r \) such that the columns of each of these matrices generate the module \( M_r \) (note that in the last three expressions, addition and multiplication under the summation sign is componentwise modulo \( m \)); \( \xi_i (X_r) \cdot (\xi_i^{(0)} (X_r)) \) is the indicator of an event \( \{ a_{i,j} X_r = 0 \} \) (\( a_{i,j}^0 X_r = 0 \)), \( i = \overline{1,n-s}, |T| \) is the cardinality of the set \( T; \)

885