Quantum groups of fermionic algebras *)

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We consider inhomogeneous quantum groups that transform various types of fermions: standard fermions, commuting fermions and orthofermions. These quantum groups are not $q$-deformations.

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1 Introduction

Symmetry transformations based on Lie groups and Lie algebras are most known in all areas of physics. However, complicated problems have required different generalizations and further development conception of symmetry.

Theory of quantum integrable systems has initiated a new type of symmetry and mathematical objects called quantum groups. Quantum groups are algebraic objects and they contain Lie groups and Lie algebras. The procedure of derivation of quantum group from Lie groups is similar to the well-known procedure of quantization of classical systems, [1].

One of the application areas of quantum group theory is quantum field theory. Quantum field theory, which describes the behavior of elementary particles and fields, basically depends on the bosonic oscillator which is described by the algebraic relation

$$cc^* - c^*c = 1$$

and the fermionic oscillator is defined by

$$cc^* + c^*c = 1,$$  \hspace{1cm} (2)
$$c^2 = 0.$$  \hspace{1cm} (3)

The crucial property of the fermionic oscillator is the Pauli exclusion principle. According to the Pauli exclusion principle the two fermions can not be in the same state. This is the meaning of $c^2 = 0$.

Another important property is that unlike the bosonic oscillator there is no standard classical analogue for the fermionic oscillators. This means that we can not talk about Bohr’s correspondence principle for the fermionic oscillators. These two properties make the fermion algebras more important than the bosonic algebra.

Since one of the application areas of the quantum groups is quantum field theory, searching quantum groups, which leave particle algebras, especially fermion algebras invariant, is important.


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2 Quantum group FIO(2d)

For \(d\) fermions, the fermion algebra can be written as

\[
\begin{align*}
 c_i c_j + c_j c_i &= 0, \\
 c_i c_j^* + c_j^* c_i &= \delta_{ij}, \quad i = j = 1, 2, \ldots, d.
\end{align*}
\]

The creation and annihilation operators are transformed via

\[
\begin{pmatrix}
 c_i \\
 c_i^* \\
 1
\end{pmatrix}' = \begin{pmatrix}
 \alpha_{ik} & \beta_{ik} & \gamma_i \\
 \beta_{ik}^* & \alpha_{ik}^* & \gamma_i^* \\
 0 & 0 & 1
\end{pmatrix} \otimes \begin{pmatrix}
 c_k \\
 c_k^* \\
 1
\end{pmatrix}.
\]  

(6)

In shorthand notation one can write equation (6) as

\[
C' = T \otimes C,
\]

(7)

where

\[
T = \begin{pmatrix}
 \alpha_{ik} & \beta_{ik} & \gamma_i \\
 \beta_{ik}^* & \alpha_{ik}^* & \gamma_i^* \\
 0 & 0 & 1
\end{pmatrix} = \begin{bmatrix}
 A & F \\
 0 & 1
\end{bmatrix}.
\]

(8)

Here, the minimal assumption is that the homogeneous parameters \(\alpha_{ik}, \beta_{ik}, \alpha_{ik}^*, \) and \(\beta_{ik}^*\) commute among themselves. To require the transformation to be an algebra homomorphism the consistency of the transformations yields the relations:

\[
\begin{align*}
\{\alpha_{ik}, \gamma_j\} &= 0, \quad \{\beta_{ik}, \gamma_j\} = 0, \quad \alpha_{ik} \beta_{jk} + \beta_{ik} \alpha_{jk} + \{\gamma_i, \gamma_j\} = 0, \\
\{\alpha_{ik}, \gamma_j^*\} &= 0, \quad \{\beta_{ik}, \gamma_j^*\} = 0, \quad \alpha_{ik} \alpha_{jk}^* + \beta_{ik} \beta_{jk}^* + \{\gamma_i, \gamma_j^*\} = \delta_{ij}.
\end{align*}
\]

(9)

together with the \(*\)-conjugates of the all relations. Hence the coproduct is defined as \(\Delta(T) = T \otimes T\). Also the antipode \(S : T = T^{-1}\) is an anti-algebra and the counit is given as \(\varepsilon(T) = I\). These operations define matrix quantum group. The inhomogeneous matrix quantum group is called Fermionic Inhomogeneous Orthogonal quantum group (FIO(2d)), [2–5].

3 Quantum group CoFI(2d)

Normally the creation (or annihilation) operators for two different fermion states are taken to be anticommuting. However, this is not necessary. As long as a single fermion creation (or annihilation) operator is taken to satisfy \(c_i^2 = (c_i^*)^2 = 0\), the Pauli exclusion principle is satisfied.

There is a simple relation between Heisenberg spin algebra and commuting fermions as

\[
\sigma_x^i = c_i + c_i^*, \quad \sigma_y^i = -i(c_i - c_i^*), \quad \sigma_z^i = c_i c_i^* - c_i^* c_i.
\]

(10)

In order to define spin operators at different sites, we should define creation and annihilation operators acting at these different sites. This necessity gives us discrete