Abstract. The construction of the extended double cover was introduced by N. Alon [1] in 1986. For a simple graph $G$ with vertex set $V = \{v_1, v_2, \ldots, v_n\}$, the extended double cover of $G$, denoted $G^*$, is the bipartite graph with bipartition $(X, Y)$ where $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$, in which $x_i$ and $y_j$ are adjacent iff $i = j$ or $v_i$ and $v_j$ are adjacent in $G$.

In this paper we obtain formulas for the characteristic polynomial and the spectrum of $G^*$ in terms of the corresponding information of $G$. Three formulas are derived for the number of spanning trees in $G^*$ for a connected regular graph $G$. We show that while the extended double covers of cospectral graphs are cospectral, the converse does not hold. Some results on the spectra of the $n$th iterated double cover are also presented.

Keywords: characteristic polynomial of graph, graph spectra, extended double cover of graph

MSC 2000: 05C50, 05C30

1. Introduction

The spectra of graphs have long been studied and the study in this field has found applications in a variety of problems in theoretical chemistry, quantum mechanics, statistical physics, computer and information sciences, as well as some areas of mathematics including spectral Riemannian geometry (see [2], [4]–[7], [9]–[11] and the cited references there).

For studying networks N. Alon [1] introduced, in 1986, the extended double cover of a graph to obtain expanders from magnifiers. This motivated our interest in studying the spectra of the extended double cover graphs.

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Throughout the paper $G$ is always used to denote a simple graph with $n \geq 1$ vertices. For a simple graph $G$ with vertex set $V = \{v_1, v_2, \ldots, v_n\}$, the extended double cover of $G$, denoted as $G^*$, is the bipartite graph with bipartition $(X, Y)$ where $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$, in which $x_i$ and $y_j$ are adjacent iff $i = j$ or $v_i$ and $v_j$ are adjacent in $G$.

For example, the complete bipartite graph $K_{n,n}$ is the extended double cover of the complete graph $K_n$. It is easy to see that $G^*$ is connected iff $G$ is connected, and $G^*$ is regular of degree $r + 1$ iff $G$ is regular of degree $r$.

For a graph $G$ with adjacency matrix $A$, the characteristic polynomial of $G$ is 
\[ \chi(G, \lambda) = \lambda^n - \alpha_1 \lambda^{n-1} + \cdots - \alpha_n, \]
where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the eigenvalues of $A$ (i.e. the zeros of $\chi(G, \lambda)$) and the spectrum of $A$ (which consists of the $n$ eigenvalues) are called the eigenvalues and the spectrum of $G$, respectively. For other notation and terminology not defined here the reader may refer to the books [2] and [3].

In the next section we shall give formulas for the characteristic polynomial and the spectrum of $G^*$ in terms of the corresponding information of $G$. Three formulas are derived for the number of spanning trees in $G^*$ for a connected regular graph $G$.

While the extended double covers of cospectral graphs are cospectral, we show the converse does not hold. Some results on the spectra of the $n$th iterated double cover are also presented.

2. Results

**Theorem 1.**

(i) \[ \chi(G^*, \lambda) = (-1)^n \chi(G, \lambda - 1) \chi(G, -\lambda - 1). \]

(ii) Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the spectrum of $G$.

Then the spectrum of $G^*$ consists of $\pm (\lambda_1 + 1), \pm (\lambda_2 + 1), \ldots, \pm (\lambda_n + 1)$.

**Proof.** Let $G$ have the adjacency matrix $A$. Then it is not difficult to see that the partitioned matrix \[
\begin{pmatrix}
0 & A + I \\
A + I & 0
\end{pmatrix}
\] is the adjacency matrix of $G^*$, in which all 0, $A$ and $I$ are $n \times n$ matrices. So, \[
\chi(G^*, \lambda) = \begin{vmatrix}
\lambda I & -(A + I) \\
-(A + I) & \lambda I
\end{vmatrix} = \lambda^n \begin{vmatrix} I & -\lambda^{-1}(A + I) \\
-(A + I) & \lambda I
\end{vmatrix}.
\]

It is well known in matrix theory (see, for example, [8, p. 45]) that if $M$ is an invertible matrix then \[
\begin{vmatrix} M & N \\
P & Q \end{vmatrix} = |M| \cdot |Q - PM^{-1}N|.
\]

So, \[
\chi(G^*, \lambda) = \lambda^n |I - \lambda^{-1}(A + I)^2| = |\lambda^2 I - (A + I)^2| = |\lambda I - (A + I)| \cdot |\lambda I + (A + I)| = (-1)^n |(\lambda - 1)I - A| \cdot |(-\lambda - 1)I - A| = (-1)^n \chi(G, \lambda - 1) \chi(G, -\lambda - 1).
\]