THE KURZWEIL-HENSTOCK THEORY OF
STOCHASTIC INTEGRATION

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Abstract. The Kurzweil-Henstock approach has been successful in giving an alternative
definition to the classical Itô integral, and a simpler and more direct proof of the Itô Formula.
The main advantage of this approach lies in its explicitness in defining the integral, thereby
reducing the technicalities of the classical stochastic calculus. In this note, we give a unified
theory of stochastic integration using the Kurzweil-Henstock approach, using the more
general martingale as the integrator. We derive Henstock’s Lemmas, absolute continuity
property of the primitive process, integrability of stochastic processes and convergence
theorems for the Kurzweil-Henstock stochastic integrals. These properties are well-known
in the classical (non-stochastic) integration theory but have not been explicitly derived in
the classical stochastic integration.

Keywords: stochastic integral, Kurzweil-Henstock, convergence theorem

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1. Introduction

Stochastic calculus has been well developed in the study of stochastic integrals,
see [3], [8], [14], [18], [19], [20], [21], [22], [32]. In the classical theory of integration
the Riemann integral with uniform mesh is found to be deficient. In the 1950s,
J. Kurzweil and R. Henstock independently modified the Riemann integral by using
non-uniform meshes, that is, meshes that vary from point to point. It turns out that
this integral is more general than the classical Riemann integral and the Lebesgue
integral, see [4], [5], [6], [9], [10], [11].

Along this line of thought, the Henstock approach, also known as the generalized
Riemann approach, has been used to study stochastic integrals, see [2], [7], [10], [12],
[13], [15], [16], [17], [23], [24], [25], [26], [27], [31], [33]. The advantage of the
generalized Riemann approach is that it gives an explicit and intuitive definition of the
stochastic integral using $L^2$-convergence. Even for the stochastic integrals, it turns out that Henstock’s definition encompasses the classical stochastic integrals, see [23], [24], [25], [26], [27]. The Henstock approach was also used to characterize stochastic integrable processes in [29] and an integration-by-part formula is also derived for stochastic integrals, see [28]. The Henstock approach has also been shown to be able to give an easier and more direct proof of the Itô Formula, see [30].

In this note, we shall establish the results of the theory of stochastic integration using the Kurzweil-Henstock approach. We shall use an $L^2$-martingale as the integrator in most of our discussion. The Kurzweil-Henstock approach is well-known for its explicitness for the classical integration theory. In addition, we also establish the convergence theorems for the stochastic integrals. As Brownian motions are special cases of $L^2$-martingales, the results of this paper encompass Itô’s stochastic integration as well (which consider Brownian motion as the integrator).

2. Setting and definition of the integral

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and \{\mathcal{F}_t\} an increasing family of $\sigma$-subfields of $\mathcal{F}$ for $t \in [a, b]$, that is, $\mathcal{F}_r \subset \mathcal{F}_s$ for $a \leq r < s \leq b$ with $\mathcal{F}_b = \mathcal{F}$. The probability space together with its family of increasing $\sigma$-subfields is called a standard filtering space and denoted by $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$.

A process \{\varphi(t, \omega) : t \in [a, b]\} on $(\Omega, \mathcal{F}, P)$ is a family of $\mathcal{F}$-measurable functions (which are called random variables) on $(\Omega, \mathcal{F}, P)$. We also denote the process $\varphi(t, \omega)$ by $\varphi_t(\omega)$.

The process \{\varphi_t(\omega) : t \in [a, b]\} is said to be adapted to the filtering \{\mathcal{F}_t\} if for each $t \in [a, b]$, $\varphi_t$ is $\mathcal{F}_t$-measurable. In this paper, we shall fix the standard filtering space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ and simplify it to the process $\varphi$ being adapted.

**Definition 1.** A measurable process $X = \{X(t, \omega), t \in [a, b]\}$ defined on $[a, b]$ is called a martingale if

(i) $X$ is adapted to the filtration, that is, $X_t$ is $\mathcal{F}_t$-measurable for each $t \in [a, b]$;
(ii) $\int_{\Omega} |X_t| \, dP < \infty$ for almost all $t \in [a, b]$; and
(iii) $E(X_t | \mathcal{F}_s) = X_s$ for all $t \geq s$.

For a given random variable $\varphi$ on $(\Omega, \mathcal{F}, P)$, let $E(\varphi)$ denote its expectation in the probability space, that is,

$$E(\varphi) = \int_{\Omega} \varphi \, dP,$$