UNIT GROUPS OF GROUP ALGEBRAS OF SOME SMALL GROUPS

GAOHUA TANG, YANGJIANG WEI, NANNING, YUANLIN LI, ST. CATHARINES

(Received November 20, 2012)

Abstract. Let $FG$ be a group algebra of a group $G$ over a field $F$ and $\mathcal{U}(FG)$ the unit group of $FG$. It is a classical question to determine the structure of the unit group of the group algebra of a finite group over a finite field. In this article, the structure of the unit group of the group algebra of the non-abelian group $G$ with order 21 over any finite field of characteristic 3 is established. We also characterize the structure of the unit group of $FA_4$ over any finite field of characteristic 3 and the structure of the unit group of $FQ_{12}$ over any finite field of characteristic 2, where $Q_{12} = \langle x, y; x^6 = 1, y^2 = x^3, xy = x^{-1} \rangle$.

Keywords: group ring; unit group; augmentation ideal; Jacobson radical

MSC 2010: 16S34, 16U60, 20C05

1. Introduction and notations

Let $FG$ be a group algebra of a group $G$ over a field $F$ and $\mathcal{U}(FG)$ the unit group of the group algebra $FG$. It is a classical question to determine the structure of the unit group of the group algebra of a finite group over a finite field. Recently there are quite a few papers which characterize the structures of unit groups of group algebras of certain small groups over finite fields (see for example [3], [5], [4], [6], [7], [9], [8], [10], [11], [15], [2], [16]). Most recently, in [2] Tang et al. determined the structures of unit groups of group algebras $FG$ of any groups of order 21 over finite fields except for the case when $G$ is the non-abelian group of order 21 and $F$ is a field of characteristic 3. The first goal of this paper is to study this remaining case. We shall determine the structure of the Jacobson radical for this group algebra and then establish the structure of its unit group.

This research was supported in part by a Discovery Grant from the Natural Sciences and Engineering Research Council of Canada, the National Science Foundation of China (11161006, 11171142), the Guangxi Natural Science Foundation (2011GXNSFA018139) and the Guangxi New Century 1000 Talents Project.
There are three non-abelian groups of order 12: $A_4$, $D_{12}$ and $Q_{12}$. In 2007, R. Sharma, J. Srivastava and M. Khan ([15]) characterized the unit group of $F A_4$ over a finite field. In the case when the characteristic of $F$ is 2 or 3, they provided only a preliminary description of the unit group. In [8], J. Gildea established a complete characterization of the unit group of $F A_4$ over a finite field of characteristic 2. In [11], Gildea and Monaghan established the structure of unit groups of $F D_{12}$ and $F Q_{12}$ over a finite field of characteristic 3. Our second goal is to determine the structure of the group algebra $F A_4$ over a finite field of characteristic 3 and establish a complete characterization of the unit group of this group algebra. In 2011, Tang and Gao ([16]) described the structure of the unit group of the group algebra $F Q_{12}$. We shall determine the structure of the Jacobson radical of $F Q_{12}$ over a finite field of characteristic 2 and provide a better characterization of the unit group of $F Q_{12}$. We note that other unit groups of group algebras of the groups of order 12 have been completely characterized (see [11], [15], [16] for details).

Throughout this paper, $A_4$ denotes the alternating group of degree 4, $Q_{12} = \langle x, y; x^6 = 1, y^2 = x^3, x^y = x^{-1} \rangle$, $C_n$ denotes the cyclic group of order $n$, $F$ is a finite field of characteristic $p$, and $F^*$ is the multiplicative group of $F$. We also denote by $M(n, F)$ and $GL(n, F)$ the ring of all $n \times n$ matrices over a field $F$ and the general linear group of degree $n$ over a field $F$, respectively. Denote by $Z(FG)$ the center of $FG$.

Recall that the ring homomorphism $\varepsilon: FG \to F$ given by

$$\varepsilon \left( \sum_{g \in G} a_g g \right) = \sum_{g \in G} a_g$$

is called the augmentation mapping of $FG$ and its kernel, denoted by $\Delta(G)$, is called the augmentation ideal of $FG$. For a subgroup $H$ of $G$, we shall denote by $\Delta(G, H)$ the left ideal of $FG$ generated by the set $\{h - 1; \ h \in H\}$. That is,

$$\Delta(G, H) = \left\{ \sum_{h \in H} \alpha_h (h - 1); \ \alpha_h \in FG \right\}.$$

If $H$ is a normal subgroup of $G$, then $\Delta(G, H)$ is a two-sided ideal. Note that the ideal $\Delta(G, G)$ coincides with the ideal $\Delta(G)$. 

150