Regular partitions of (weak) finite generalized polygons

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Abstract In this paper we define a regular m-partition of a distance regular graph as a partition of the vertex set into m classes, such that the number of vertices of a given class adjacent to a fixed vertex of another class (but possibly the same), is independent of the choice of that vertex in this class. Furthermore, we exhibit a technique to determine exact, discrete or bounding values for the intersection numbers of two such regular partitions of a DRG. As an application, we perform a structural investigation on the substructures of finite generalized polygons and, besides some new results, we give unifying, alternative and more elegant proofs of the results in Offer (J Combin Theory Ser A 97: 184–186, 2002) and Offer (Discrete Math 294: 147–160, 2005).

Keywords Distance regular graphs · Regular partition · Generalized polygons

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1 Introduction

The present paper stems from an observation made in the thesis [5] of the first author, where she notes that, in any finite generalized hexagon of order s, the intersection of any distance-2 ovoid with an arbitrary distance-3 ovoid (if these exist) is a constant only depending on s (namely, $s^2 - s + 1$). This observation used the orthogonality of eigenvectors belonging to distinct eigenvalues of the adjacency matrix of the point graph of the generalized hexagon. Soon it became clear that this simple “trick” was not yet fully exploited in the literature,
and that it has a lot of other applications. In the present paper, we will apply it systematically to finite (weak) generalized polygons. To that end, we have to describe and introduce the technique not only for generalized hexagons, but for generalized polygons in general. It turns out that we can then easily phrase everything in terms of distance regular graphs (weak generalized polygons with an order are examples of these) and that is exactly what we shall do. However, we will only look at weak generalized polygons for applications.

Concerning the applications, we motivate our study as follows. There has been a recent ongoing and growing interest in ovoids, partial ovoids, coverings and blocking sets of all kind of subspaces of polar and projective spaces, especially focusing on bounds, existence, and non-existence. These investigations mainly come from problems in projective planes and in generalized quadrangles, where these objects have proved very useful and important. In this paper, we want to study a generalization in the other direction: instead of looking at higher rank geometries (generalized quadrangles are polar spaces of rank 2), we take a look at larger diameter (generalized quadrangles are the generalized polygons of diameter 2). In fact, this study was initiated in Chap. 7 of [17], where the notion of distance-\(j\) ovoid was introduced in full generality, as possible ways to generalize the notion of ovoid in a generalized quadrangle. In the meantime, distance-\(j\) ovoids have proved to be useful objects with applications in the theory of perfect codes and two-weight codes, for instance. So the present paper lays the foundations for further study of bounds, existence, non-existence, classification, and characterization of special point sets in finite weak generalized polygons.

Concerning the type of point subsets we will consider, we motivate this as follows. Distance-\(j\) ovoids play a central role as these have important applications, see above. Despite the fact that ovoids in quadrangles generalize to both distance-2 and distance-3 ovoids in hexagons, there is another generalization, to so-called spheres, in hexagons. These arise when considering, in a generalized hexagon of order \((s, s^3)\), the set of points of a subhexagon of order \(s\) subtended by a point not on a line of this subhexagon; see [3], where this idea is used to prove a characterization of the twisted triality hexagon \(T(q, q^3)\) and its dual. So we include these spheres into our results. Also, when dealing with groups and homogeneous subsets of points, subgroups with few orbits usually give interesting examples of such sets. In particular, the existence of a large stabilizer implies certain regularity properties of the point set in question. These properties are included in our axioms for regular partitions, in particular in the definition of regular partial ovoids (however, we do not look at the consequences of our results to possible classification results using groups; this will be done elsewhere).

Besides many new results (among which those that come directly from [5]), we include all the non-existence results of [11,12], which have much shorter proofs in our setting. We also provide some more intersection properties of the objects (floveads) introduced in [12].

It is worth noting that our technique is not only useful for proving non-existence of certain objects, and intersection properties of distinct objects, but also to prove existence. Indeed, we prove the existence of a regular partial ovoid using in a crucial way the intersection properties of ovoids with spheres; see Example 4.15 below.

The paper is organized as follows. In Sect. 2, we introduce the various notions. In Sect. 3, we explain our technique in general for distance regular graphs. And, finally, in Sect. 4 we apply the technique to a lot of substructures of finite weak generalized polygons having an order.