Note on the existence of translation nets

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Abstract We exhibit a class of \((2^k + 1, 2^k)\)-translation nets with nonabelian translation group, for any natural \(k\). At the same time, it is the first infinite class of translation nets known to admit nonisomorphic translation groups for each of its elements.

Keywords Translation net · Nonisomorphic translation groups · Generalized quadrangle

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1 Introduction

A translation net \(\mathcal{N}\) is a net for which there is a group \(G\) of automorphisms of \(\mathcal{N}\) each element of which fixes each parallel class of \(\mathcal{N}\), and so that \(G\) acts sharply transitively on the points of \(\mathcal{N}\). (If \(\mathcal{N}\) has order \(k\) and degree \(r\), \(k, r \geq 2\), and if \(k = r - 1\), then \(\mathcal{N}\) is a translation plane.) From a translation net of order \(s\) and degree \(r \geq 3\) one can construct an \((s, r)\)-PCP (see, e.g., [1] for a definition of the latter), and the opposite direction is also true [1].

Quite in contrast to the theory of translation planes, translation groups of translation nets need not be abelian. In [1], the first infinite series of large translation nets are constructed with nonabelian translation group. (We refer the reader to [1] for a discussion on the term “large”.)

Also, in [1] \((q^2, q + 1)\)-PCPs are constructed for \(q\) any odd prime power, for \(q = 2^n\) with \(n\) not a power of 2, and for \(q \geq 8\) an odd power of 2. No other \((q^2, q + 1)\)-PCPs with \(q\) a power of 2 are known.
In this note, we aim at constructing translation nets $N$ of order $q^2$ and degree $q + 1$ with nonabelian translation groups $G$ for $q$ any power of 2, thus completing the series of examples with parameters $(q^2, q + 1)$, $q$ a prime power. These nets also provide the first infinite class of examples of translation nets which admit nonisomorphic translation groups. Only a few of such nets seem to be known - see [4].

We will address the theory of generalized quadrangles (GQs) for our purpose. Standard information can be found in [2], or the more recent reference [6].

2 Construction

The following can be found in [2,1.3.1].

Let $p$ be a regular point of a GQ $S = (P, B, I)$ of order $(s, t)$, $s \neq 1 \neq t$. Then the incidence structure with point set $p_\perp \setminus \{p\}$, with line set the set of spans $\{q, r\}_\perp \perp$, where $q$ and $r$ are noncollinear points of $p_\perp \setminus \{p\}$, and with the natural incidence, is the dual of a net $N_p$ of order $s$ and degree $t + 1$.

If in particular $s = t$, there arises a dual affine plane of order $s$. (Also, in the case $s = t$, the incidence structure $\pi_p$ with point set $p_\perp$, with line set the set of spans $\{q, r\}_{\perp \perp}$, where $q$ and $r$ are different points in $p^\perp$, and with the natural incidence, is a projective plane of order $s$.)

Suppose that $S$, still having a regular point $p$, admits an automorphism group $H$ that fixes each line incident with $p$ while acting sharply transitively on the points opposite $p$. (In other words, $p$ is an elation point with elation group $H$.) Then it can be shown (see, e.g., [7]) that $H$ contains a subgroup $K$ of order $t$ that acts trivially on $p_\perp$. Then clearly $N_p$ is a translation net with parameters $(t + 1, s)$, the translation group being naturally induced by $H/K$.

Now let $H$ be a nonsingular Hermitian variety in the projective space $\text{PG}(3, q^2)$. The points and lines of $H$ form a generalized quadrangle $H(3, q^2)$, which has order $(q^2, q)$, all of whose points are regular elation points [2]. Note that the automorphism group of $H(3, q^2)$ acts transitively on the points of the quadrangle. Fix some point $x$. If $q$ is odd, it can be shown that there is a unique elation group $H$ w.r.t. this point [5]. Moreover, $H/K$ (using the notation of above) is elementary abelian. When $q$ is even the situation is entirely different: there are precisely two classes of nonisomorphic elation groups w.r.t. $x$, say $C_1$ and $C_2$, with the following properties.

- $|C_1| = 1$ and $|C_2| = q^2 - 1$.
- For each $R \in C_1 \cup C_2$, $K = Z(R)$ (the center of $R$).
- Elements of $C_1$ are of nilpotency class 2, while elements of $C_2$ have class 3.

(All relevant information can be found in [3,5].) So we can conclude that if $R \in C_1$, $R/K$ is an (even elementary) abelian group for $N_x$, and if $R \in C_2$, $R/K$ is of class 2.

This completes the proofs of the results announced in the abstract. □

Remark 2.1 (i) The net $N_x$ arising as above is well-known. Its dual is the dual net $H_q^3$, $n > 2$, constructed as follows: the points of $H_q^3$ are the points of $\text{PG}(3, q)$ not on a given line $\text{PG}(1, q) \subset \text{PG}(3, q)$; the lines of $H_q^3$ are the lines of $\text{PG}(3, q)$ which have no point in common with $\text{PG}(1, q)$; the incidence in $H_q^3$ is the natural one.