On quasi-symmetric designs with intersection difference three

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Abstract In a recent paper, Pawale (Des Codes Cryptogr, 2010) investigated quasi-symmetric 2-(v, k, \( \lambda \)) designs with intersection numbers \( x > 0 \) and \( y = x + 2 \) with \( \lambda > 1 \) and showed that under these conditions either \( \lambda = x + 1 \) or \( \lambda = x + 2 \), or \( \mathcal{D} \) is a design with parameters given in the form of an explicit table, or the complement of one of these designs. In this paper, quasi-symmetric designs with \( y = x + 3 \) are investigated. It is shown that such a design or its complement has parameter set which is one of finitely many which are listed explicitly or \( \lambda \leq x + 4 \) or \( 0 \leq x \leq 1 \) or the pair \((\lambda, x)\) is one of \((7, 2), (8, 2), (9, 2), (10, 2), (8, 3), (9, 3), (9, 4)\) and \((10, 5)\). It is also shown that there are no triangle-free quasi-symmetric designs with positive intersection numbers \( x \) and \( y \) with \( y = x + 3 \).

Keywords Quasi-symmetric designs · Strongly regular graphs · Triangle-free design

Mathematics Subject Classification (2000) 05B05

1 Introduction

Let \( \mathcal{D} \) be a 2-(v, k, \( \lambda \)) design. Here as usual, \( v \) denotes the number of points of \( \mathcal{D} \), \( k \) the block size and \( \lambda \) the number of occurrences of pairs of points in the blocks of \( \mathcal{D} \). Then each point
occurs in a constant number \( r \) of blocks of \( D \). If \( b \) denotes the number of blocks of \( D \), then the parameters \((v, k, \lambda, r, b)\) satisfy the basic relations \( bk = vr, \lambda(v - 1) = r(k - 1)\), and Fisher’s inequality \( b \geq v \).

For general notation and concepts in design theory, we refer to Beth, Jungnickel, and Lenz [2] or Hughes and Piper [7]. A design with \( v = b \) (equivalently \( r = k \)) is known as a symmetric \( 2-(v, k, \lambda) \)-design. The intersection numbers of \( 2-(v, k, \lambda) \)-design are the cardinalities of the intersection of any two distinct blocks. It is well known that a \( 2-(v, k, \lambda) \)-design is symmetric if and only if \( \lambda = 1 \). Let \( x \) and \( y \) be non-negative integers with \( x \leq y < k \). A design \( D \) is called quasi-symmetric with intersection numbers \( x \) and \( y \) if any two distinct blocks of \( D \) intersect in \( x \) or \( y \) points and both intersection numbers are realized. We refer to Shrikhande and Sane [26] as a basic reference on quasi-symmetric designs. A quasi-symmetric design is called proper if \( x \neq y \) and improper otherwise. Clearly symmetric designs are improper quasi-symmetric designs and any \( 2-(v, k, 1) \) design with \( b > v \) is a proper quasi-symmetric design with \( x = 0 \) and \( y = 1 \). Thus linear spaces, that is \( 2-(v, k, 1) \) designs, give examples of proper and improper quasi-symmetric designs.

A \( 2-(v, k, \lambda) \) design is called resolvable if its blocks can be partitioned in subsets called parallel classes such that each parallel class partitions the point set. A partition of the blocks is called a parallelism with blocks in the same class being parallel. Two distinct parallel blocks are disjoint. If further, any two blocks from different parallel classes intersect in a constant number \( y \) (say) of points, the design is called affine. Affine designs are thus quasi-symmetric with \( x = 0 \) and \( y \).

Examples of quasi-symmetric designs which are not symmetric, or affine designs, or linear spaces are rather rare, so construction methods of quasi-symmetric designs are of interest. The problem of classifying quasi-symmetric 2-designs, even for the case \( x = 0 \) appears to be a difficult open problem. As a consequence, one approach in the study of such designs has been to put additional parametric or structural restrictions. Baartmans and Shrikhande [1]; Limaye, Sane, and Shrikhande [11]; Mavron and Shrikhande [13]; Cameron [6]; Sane and Shrikhande [24]; McDonough and Mavron [16]; Mavron, McDonough and Shrikhande [14] are some papers where additional structural conditions are imposed. Pawale [21] studies quasi-symmetric 2-designs satisfying a parametric condition of the form \( y - x \) has a fixed value.

In a recent preprint, Pawale [22] obtained a parametric classification of proper quasi-symmetric 2-designs with \( y - x = 2 \), with \( x > 0 \) and \( \lambda > 1 \). It is shown in [22] that if \( D \) is a quasi-symmetric 2-design with these conditions, then either \( \lambda = x + 1 \) or \( \lambda = x + 2 \), or \( D \) is a design with parameters given in the form of an explicit table, or the complement of one of these designs.

Suppose now that \( y = x + 3 \) in a proper quasi-symmetric 2-design. The following are the currently known examples of such designs: The affine \( 2 - (27, 9, 4) \) designs considered by Lam and Tonchev [10] with \( x = 0 \) and \( y = 3 \); the geometric \( 2-(121, 13, 13) \) design \( D = PG_2(4, 3) \) of points and planes of the projective space \( PG(4, 3) \), with \( x = 1 \) and \( y = 4 \); a non-geometric \( 2-(121, 13, 13) \) design with \( x = 1 \), \( y = 4 \) given in Jungnickel and Tonchev [8], which is a special case of an infinite class of quasi-symmetric designs with \( x = 1 \), \( y = q + 1 \), where \( q \) is a prime power, with parameters the same as those of the geometric design \( PG_d(2d, q) \); a class of quasi-symmetric \( 2-(66, 30, 29) \) designs with \( x = 12 \) and \( y = 15 \) and a class of quasi-symmetric \( 2-(78, 36, 30) \) designs with \( x = 15 \) and \( y = 18 \) constructed by Bracken, McGuire and Ward [3]. See also McDonough, Mavron and Ward [17] for an alternative description of quasi-symmetric designs with these parameters.