ABSTRACT. The definition of ‘definition’ cannot be taken for granted. The problem has been treated from various angles in different journals. Among other questions raised on the subject we find: the notions of concept definition and concept image, conceptions of mathematical definitions, redefinitions, and from a more axiomatic point of view, how to construct definitions. This paper will deal with ‘definition construction processes’ and aims more specifically at proposing a new approach to the study of the formation of mathematical concepts. I shall demonstrate that the study of the defining and concept formation processes demands the setting up of a general theoretical framework. I shall propose such a tool characterizing classical points of view of mathematical definitions as well as analyzing the dialectic involving definition construction and concept formation. In that perspective, a didactical exemplification will also be presented.

KEY WORDS: concept formation, definition, discrete straight line, processes of definition construction

1. INTRODUCTION

There is a far cry between the situation of the student striving to grasp a new mathematical concept and the neat definition produced by a mature professional mathematician. To understand how concept formation works implies exploring the wide field of mathematical definitions considered as concept holders. How can we reconcile rigour and clarity with the universe of trial and error, misdirected moves jostling with sudden insights in which the teacher and students labour? It is precisely in those arduous moments though, that our students need our help most.

So as to map the terrain provisionally with definitions serving as temporary markers for concept formation, we have, therefore to work out a theoretical framework through empirical research. Although definition construction has admittedly a place in mathematical research, precious little has been written on the subject in specialised journals. However, several features of definition are commonly accepted as crucial (Zaslavsky-Shir, 2005).¹ A frequently used approach to definitions leads us to consider that a clear definition is a part of a theory (Mariotti-Fischbein, 1997).
Bearing in mind the aforesaid situation, I note: Lakatos (1961) worked simultaneously on concept formation and definition construction: “A definitional procedure is a procedure of concept formation” (p. 54). Lakatos contributed to the debate on formulating a model of mathematical discovery while integrating both the social and the conceptual aspects.

Freudenthal (1973), Mariotti and Fischbein (1997) and Borasi (1992) have pointed to some didactical situations involving defining processes in geometry. The theoretical tools mobilized by them did not focus on the definitional procedure itself. Moreover, classification and redefining tasks are actually only the tip of the iceberg consisting of Situation(s) of Definition Construction (called from now on SDC(s)).

The proposed theoretical framework will extract tools from existing didactical and cognitive theories (e.g., Vergnaud, 1991, and the theory of conceptual fields; Vinner, 1991, and the concept image). This will lead me to a characterisation of Schoenfeld’s “problem-solving heuristics” (1987, p. 18). I shall adopt a pragmatic position about “problem” and “problem-solving”, and focus on the characterisation of situations so as to diagnose students’ heuristics. This approach will be boosted by an analysis of historical and present uses of mathematics (e.g. D’Ambrosio, 1993; Lave, 1988; Nuñes et al., 1983; Sierpinska, 1989; Thurston, 1994).

Tall (2004, p. 287) gains “an overview of the full range of mathematical cognitive development” by scanning a whole range of theories. A global vision of mathematical growth then emerges, making room for three worlds of thinking: the “embodied world”, the “proceptual world” and the “formal world”. In this way, a more coherent view of cognitive development may be obtained. Endorsing this point of view, I will question the place of definitions in such a theory. “Formal definitions” admittedly belong to Tall’s “formal world”. What happened before the “smooth” definitions were arrived at? What were the heuristic processes involved? Although the apprehension of new mathematical concepts began in the “embodied world” through perception, I still assume that the “proceptual world” is not always adequate to characterize a concept which is being constructed. So how are we going to grasp the dialectic between concept formation and definition construction within this theoretical range? I think we can safely assume that there is another world, different from the “embodied”, “proceptual” and “formal” worlds, which is both transversal and complementary, fostering the characterisation of mathematical growth through definition construction processes in particular.