Ellipsoidal Domains: Piecewise Nonuniform and Impotent Eigenstrain Fields

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Abstract In association with multi-inhomogeneity problems, a special class of eigenstrains is discovered to give rise to disturbance stresses of interesting nature. Some previously unnoticed properties of Eshelby’s tensors prove useful in this accomplishment. Consider the set of nested similar ellipsoidal domains \( \{ \Omega_1, \Omega_2, \ldots, \Omega_{N+1} \} \), which are embedded in an infinite isotropic medium. Suppose that

\[
\Omega_t = \{ x \mid x \in \mathbb{R}^3, \sum_{p=1}^{3} \frac{x_p^2}{a_p^2} \leq \xi_t^2 \},
\]

in which \( 0 \leq \xi_1 < \xi_2 < \cdots < \xi_{N+1} \) and \( \xi_t a_p, \ p = 1, 2, 3 \) are the principal half axes of \( \Omega_t \). Suppose, the distribution of eigenstrain, \( \epsilon_{ij}^{*}(x) \) over the regions \( \Gamma_t = \Omega_{t+1} - \Omega_t, \ t = 1, 2, \cdots, N \) can be expressed as

\[
\epsilon_{ij}^{*}(x) = \begin{cases} 
 f_{ijkl-m}^{(t)} \left( \sum_{p=1}^{3} \frac{x_p^2}{a_p^2} \right) x_k x_l \cdots x_m, & x \in \Gamma_t, \\
 0, & x \in \Omega_1 \cup \left( \mathbb{R}^3 - \Omega_{N+1} \right),
\end{cases}
\]

where \( x_k x_l \cdots x_m \) is of order \( n \), and \( f_{ijkl-m}^{(t)} \) represents \( 3N(n+2)(n+1) \) different piecewise continuous functions whose arguments are \( \sum_{p=1}^{3} x_p^2/a_p^2 \). The nature of the disturbance stresses due to various classes of the piecewise nonuniform distribution of eigenstrains, obtained via superpositions of Eq. (‡), is predicted and an infinite number of impotent eigenstrains are introduced. The present theory not only provides a general framework for handling a broad range of nonuniform distribution of eigenstrains, according to the set of nested domains, and is particularly useful for solving various multi-inhomogeneity problems in materials science, e.g., in the study of nanocomposites and nanostructured materials.
of eigenstrains exactly, but also has great implications in employing the equivalent inclusion method (EIM) to study the behavior of composites with functionally graded reinforcements.

Key words  ellipsoidal domain · similar ellipsoids · nonuniform eigenstrain · impotent eigenstrain · Eshelby’s tensor · 3D elastic fields · exact solution

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1 Introduction

Notions of inclusion and inhomogeneity are differentiated by Mura [1]. When a subdomain $\Omega$ of $\mathbb{R}^3$ is such that, the eigenstrain $\epsilon^*_ij(x)$ is nonzero within $\Omega$ but it is zero in $\mathbb{R}^3 - \Omega$, then $\Omega$ is called an inclusion provided that $\Omega$ and $\mathbb{R}^3 - \Omega$ have the same elastic moduli. In addition, if $\Omega$ and $\mathbb{R}^3 - \Omega$ have different elastic moduli, $\Omega$ is called an inhomogeneous inclusion. In the latter case, $\Omega$ is said to be an inhomogeneity if $\epsilon^*_ij(x) = 0$ in $\Omega$.

Consider an ellipsoidal inclusion $\Omega$ inside an infinite isotropic medium $\mathbb{R}^3$

$$\Omega = \left\{ x \mid x \in \mathbb{R}^3, \sum_{p=1}^{3} \frac{x_p^2}{a_p^2} \leq 1 \right\},$$

where $a_p$, $p = 1, 2, 3$ are the principal half axes of $\Omega$, and $x_p$ denotes the Cartesian coordinate which coincides with the corresponding axis of ellipsoid. Eshelby [2] has shown that, when the distribution of eigenstrain field over an ellipsoidal inclusion $\Omega$ is uniform, the corresponding strain and stress fields become uniform at every point inside $\Omega$. Eshelby [2] proposed equivalent inclusion method (EIM) as a treatment of inhomogeneity problems. Ever since the pioneering contributions of Eshelby [2-4], the subject of ellipsoidal inhomogeneity, due to its valuable engineering applications, became one of the most attractive problem of solid mechanics. Mura et al. [5] give a review on the inclusion problems, however, after that date there are yet numerous interesting contributions pertinent to the work of Eshelby.

Moschovidis and Mura [6] studied interacting ellipsoidal inhomogeneities by the EIM. Part of their work is devoted to a single inhomogeneity in an infinitely extended isotropic matrix. These investigators have realized that, when the applied far-field stress is linear, and the inhomogeneity is a void the consistency equations become singular. This manifests itself to existence of infinite number of so-called impotent eigenstrains in $\Omega$, whose disturbance stresses are identically zero throughout the entire medium $\mathbb{R}^3$. Subsequently based on this finding, Fruhashi and Mura [7] presented a class of explicit expression for the impotent eigenstrain associated with a single ellipsoidal inclusion. Fruhashi and Mura [7] showed that, the eigenstrains $\epsilon^*$ derived from a vector field $u^*$, which is zero along the boundary of $\Omega$ are impotent. Specifically, if $u^*$ takes on the form

$$u^*_i(x) = \begin{cases} \left( \sum_{p=1}^{3} \frac{x_p^2}{a_p^2} - 1 \right) g_i(x), & x \in \Omega, \\ 0, & x \in \mathbb{R}^3 - \Omega, \end{cases}$$

where