Graphical Illustrations for the Nur-Byerlee-Carroll Proof of the Formula for the Biot Effective Stress Coefficient in Poroelasticity

Stephen C. Cowin · Mohammed Benalla

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Abstract This work presents a graphically illustrated version of the Nur-Byerlee-Carroll proof of the formula for the Biot effective stress coefficient in poroelasticity. The original elegant proof was provided by Nur and Byerlee (J. Geophys. Res. 76:6414, 1971) for isotropic materials and extended by Carroll (J. Geophys. Res. 84:7510–7512, 1979) to anisotropic materials. Although the application of this result is in poroelasticity or in the analysis of composite materials, the proof is an analytical thought experiment in linear elasticity, and should be appreciated as such.

Keywords Poroelasticity · Anisotropy · Compressibility · Biot coefficient

Mathematics Subject Classification (2000) 74B05 · 74B99 · 74E10 · 74F10 · 74L10

1 Introduction

Poroelasticity is a theory that models the interaction of deformation and fluid flow in a fluid-saturated porous medium. The deformation of the medium influences the flow of the fluid and vice versa. The theory was proposed by Biot [3] for isotropy and Biot [4] for anisotropy as theoretical extensions of soil consolidation models developed to calculate the settlement of structures placed on fluid-saturated porous soils. The theory has been widely applied to geotechnical problems beyond soil consolidation, most notably problems in rock and bone and articular cartilage mechanics. Certain porous rocks, marbles and granites have material properties that are similar to those of bone. The effective medium (parameter) approach appears in a primitive form in the early Biot work [3] and grows in sophistication with time.

S.C. Cowin · M. Benalla
The New York Center for Biomedical Engineering & The Departments of Biomedical & Mechanical Engineering, The School of Engineering of The City College and The Graduate School of The City University of New York, New York, NY 10031, USA
e-mail: sccowin@gmail.com

M. Benalla
Mechanical Engineering and Industrial Design, New York City College of Technology, New York, USA
through the work of Nur and Byerlee [1], Rice and Cleary [5], Carroll [2], Thompson and Willis [6] and Detournay and Cheng [7]. The development of effective moduli/parameter theory over the last thirty-five years occurred almost in parallel with the increasing sophistication and refinement of the original Biot formulation. The second approach to poroelasticity, not considered here, is the mixture theory approach; it is based on diffusion models and has a different philosophy and a longer history than the effective medium approach, Truesdell [8], Truesdell and Toupin [9], Bowen [10, 11], and Coussy [12].

In this contribution the focus is on the Biot effective stress coefficient in compressible poroelasticity. In the case of isotropy the Biot effective stress coefficient is denoted by $\alpha$ and appears as a multiplicand of the pore pressure in the definition of the effective stress $T_{\text{eff}}$, $T_{\text{eff}} = T + \alpha p I$. This relationship is a part of the stress-strain-pressure relationship in poroelasticity and is, in reality, closer to an effective medium result than a poroelasticity result. The purpose of the following section is to describe the stress-strain-pore pressure relations in compressible poroelasticity as a background for the graphical illustration of the proof of this result that appears in the final section.

Terzaghi [13] suggested that $\alpha$ should be 1, which is correct only if both poroelastic constituents are incompressible. Geertsma [14] and Skempton [15] suggested that $\alpha = [1 - (K_d/K_m)]$ where $K_d$ is the drained bulk modulus and $K_m$ is the matrix material bulk modulus when the constituents are both compressible. Nur and Byerlee [1] presented a rigorous and elegant proof that the suggestion of Geertsma [14] and Skempton [15] was correct. Carroll [2] extended the Nur and Byerlee [1] proof from isotropic to anisotropic materials. For anisotropic materials the factor $T_{\text{eff}} = T + \alpha p I$ becomes a second order tensor denoted by $A$. The graphically illustrated derivation of the formula for $A$ in terms of the anisotropic material properties of the porous matrix is the objective of this contribution.

A slightly unconventional tensor-matrix notation is employed in this presentation. Its objective is to represent fourth rank tensors as matrices that are composed of tensor components, something that the classical Voigt [16] matrix notation for the anisotropic elasticity tensor does not achieve. In the notation employed here second and fourth rank tensors in three dimensions are represented as vectors and second rank tensors, respectively, in six dimensions. Transformations in the six-dimensional space, corresponding to three-dimensional transformations, are six-by-six matrix multiplications that are easily entered and quickly computed with the symbolic algebra software (Maple, Mathematica, MacSyma, and MatLab). In particular the three-dimensional fourth rank elasticity tensor is represented as a second rank tensor in a space of six dimensions. This notation has been developed and employed in a number of papers (Mehrabadi and Cowin [17], Cowin and Mehrabadi [18, 19], Mehrabadi et al. [20], Cowin and Yang [21, 22], Cowin et al. [23], Cowin [24], Cowin and Mehrabadi [25] and Cowin and Doty [26]). In particular Cowin and Mehrabadi [25] employed this notation in the development of poroelasticity that is followed in the present paper.

2 Background

In this contribution the focus is on the Biot effective stress coefficient in compressible poroelasticity. This relationship is a part of the stress-strain-pressure relationship in poroelasticity and is, in reality, closer to a composite materials result than a poroelasticity result. The purpose of this section is to describe the stress-strain-pore pressure relations in compressible poroelasticity as a background for this relationship.