A computer-assisted geometric approach to the analysis of the impact of billiard balls. Part I: Ideal impacts

Stefano Pasquero

Received: 24 March 2009 / Accepted: 5 August 2010 / Published online: 22 August 2010
© Springer Science+Business Media B.V. 2010

Abstract
It is shown how some recent general theoretical results on impacts of mechanical systems with unilateral constraints, obtained by means of Differential Geometric Impulsive Mechanics, can be operatively applied to the study of impacts between rigid bodies. The applicability of these geometric techniques is partly discussed for general impacts of rigid bodies. In particular, the general aspects of the algorithm are described and applied to analyze the impacts of two equal billiard balls moving on the plane in all possible ideal situations: when the balls can freely slide or roll on the plane of the billiard and/or between themselves. The use of symbolic-computation software is indispensable to solve the computational difficulties arising because of the high number of degrees of freedom of the system. It allows to obtain explicit expressions for the post-impact linear and angular velocities of the balls, and therefore a complete quantitative and qualitative analysis of any particular ideal impact with assigned pre-impact positions and velocities. The data regarding the usual billiard impact with the object ball at rest are explicitly listed and illustrated.

Keywords
Billiard · Impact · Impulsive reaction

1 Introduction

Geometric Impulsive Mechanics has recently given important theoretical results in Impact Mechanics, when techniques of modern Differential Geometry, and in particular jet-bundle techniques, were fruitfully applied to investigate the concept of ideality of unilateral constraints, even in the presence of kinetic constraints [1–3]. These results can be directly applied in the study of several classical impact problems, such as that of a rod and a (smooth or rough) plane, or the impact of a billiard ball rolling on a horizontal plane with a vertical (smooth or rough) wall.

The differential geometric approach to Impulsive Mechanics consists in the construction of a suitable differential-geometric context made of affine and vector spaces associated to the colliding system and its (unilateral and possibly kinetic) constraints, in a detailed analysis of the possible constitutive characterizations of the whole set of constraints acting on the system, and then in a geometrically and physically motivated choice of one of the possible constitutive characterizations. It is then by its very nature different from the classical approach (as presented for example in [4, 5]) for two main reasons: the first is the geometric framework (differentiable manifolds instead of...
Euclidean spaces); the second is that no analysis of the contact forces is required for the choice of the constitutive characterization (since geometric and energetic analyses suffice). Notwithstanding these clear differences, the geometric approach too gives significant answers, for a wide class of impulsive systems, to the problem of determining the post-impact velocities starting from the knowledge of pre-impact velocities and the geometry of the system (see once again [1–3]).

Some remarks must be underlined in order to apply the tools of Geometric Impulsive Mechanics to the problem of collision between rigid bodies. The first is that the results of Geometric Impulsive Mechanics were obtained for general systems subject to unilateral constraints with the condition that the contact between system and unilateral constraint happens in a single point. Then they can be effectively applied to a problem of collision between rigid bodies only when the contact between the bodies happens in a single point and the contact conditions can be explicitly expressed in terms of a unilateral constraint acting on the system. The simplest example of such a case is the impact of two rigid billiard balls, which can be handled as the impact of the system formed by a pair of billiard balls with a unilateral constraint obtained by fixing the distance of the centers greater than or equal to the sum of the radii.

The second remark is that collisions between rigid bodies approached with geometric techniques generally involve high-dimensional differentiable manifolds: planar impacts between two free rigid bodies is in general described (taking time into account) in geometric contexts of dimension 7 and 13, while spatial impacts between free rigid bodies is framed in geometric contexts of dimension 13 and 25. Therefore the consequent computational difficulties, when approached with hand techniques, turn out to be almost insurmountable, even for quite simple systems. The computational problems can become even worse in the presence of permanent or instantaneous kinetic constraints. Once again, as shown in this paper, the impact of billiard balls offers clear examples of the possible computational difficulties.

In this paper we present an algorithm based on the results of Geometric Impulsive Mechanics suitable for investigating collision problems between rigid bodies in all possible ideal cases. The algorithm plays a role in Geometric Impulsive Mechanics similar to that of the algorithm of Kane and Levinson [6] in the context of Classical Impact Mechanics. The main differences consist in the facts that the classical algorithm can deal also with non-ideal impacts (but a non-ideal version of the geometric algorithm will be the argument of a future paper) while the geometric algorithm can in a very simple way deal with permanent kinetic constraints acting on the system.

The aims of the paper are then three fold: (i) to show that the results of Geometric Impulsive Mechanics can be applied also to collision problems of rigid bodies and not only to systems subject to unilateral constraints; (ii) to describe the algorithm based on these results showing moreover that it can be implemented and applied to some important problems giving explicitly computed exact (and not numerical) solutions; (iii) to show that even in simple cases the computational difficulties are so big as to require the intensive use of symbolic-computation software.

We choose to pursue these aims by presenting the algorithms for the problems of the ideal impacts of two identical billiard balls. This choice is made for several reasons. Billiard games have always presented interesting challenges and severe tests for all who study Classical Impulsive Mechanics, because of their possibility of being modelled with a lower or higher degree of simplification. Then there is a wide possibility of comparison of the results of the geometric approach with those obtained with more classical ones (see for example, among the extensive literature, [7] and the references therein) and with the “real” behavior of balls in the game. A second reason is that the geometric framework of the system is so simple that it can be described without the technicalities typical of the general approach. A third one is that, although the solution algorithm remains essentially the same, the results of the various ideal situations show how the post-impact-velocity formulas can turn out to be quite simple or very complicated depending on what ideal situation is analyzed. A fourth one is that, with minor changes, the same algorithm used for the impact of billiard balls allows the study of impacts of several other systems.

We present the algorithm for the impacts of billiard balls in all the ideal cases (even those that do not have a particular physical meaning). The cases are obtained by combining in every possible way smoothness (labelled with the letter $s$) and roughness (labelled with the letter $r$) of contacts: between ball n.1 and plane, ball n.2 and the plane, balls n.1 and n.2. Then, when the initial data do not distinguish the balls, we have six different cases, while when the initial data distinguish the balls, the cases become 8. In everyone of them, the theory is powerful enough