Discussion of “Copulas: Tales and facts”,
by Thomas Mikosch
Efficient estimation of copula parameters

Johan Segers

Published online: 31 October 2006
© Springer Science + Business Media, LLC 2006

Abstract As Prof. Mikosch correctly points out, there exists very little sound statistical theory on modelling dependence using copulas. In this contribution, an open problem is presented concerning the efficient estimation of the parameter of a copula when no parametric assumptions are made regarding the marginal distributions.

Consider the semiparametric model consisting of all $d$-variate distribution functions with copula belonging to some parametric family, $\{C_\theta \mid \theta \in \Theta\}$, and with arbitrary continuous marginal distribution functions. The interest is in the copula parameter vector $\theta$. The marginal distributions are regarded upon as infinite-dimensional nuisance parameters.

The model is reasonable in case the sample size is large enough to estimate the marginals nonparametrically but not large enough to cope with the curse of dimensionality, necessitating a parametric model for the dependence. So suppose that we observe a random sample from a distribution in this model. Then how to draw inference on the unknown dependence parameter, $\theta$, in the most efficient way?

As it stands, the question is meaningless because it is too vague. We should be more specific on what kind of copula models we allow (answer: regular parametric models), which kind of estimators we allow (answer: regular estimators) and what we mean by efficient estimation (answer: minimal asymptotic


Supported by a VENI grant of The Netherlands Organization for Scientific Research (NWO).

J. Segers (✉)
Institut de Statistique, Université Catholique de Louvain, Voie du Roman Pays, 20, B-1348 Louvain-la-Neuve, Belgium
e-mail: segers@stat.ucl.ac.be
The precise definitions are somewhat technical, but intuitively, the question is clear: given the model and the data, what is the best way to estimate the copula parameter, $\theta$?

The natural candidate estimator is the pseudo-likelihood estimator proposed in Genest et al. (1995). The procedure works in two steps:

1. Estimate the marginal distribution functions by the empirical distribution functions.
2. Pretend as if the data transformed marginally by the empirical distribution functions are a random sample from the unknown copula and estimate the unknown parameter by maximum likelihood.

This is a very reasonable procedure. In the particular case of the family of bivariate Gaussian copulas indexed by a correlation parameter $-1 < \rho < 1$, the resulting estimator is nothing less but the normal scores rank correlation coefficient, also called the Van der Waerden rank correlation coefficient. In Klaassen and Wellner (1995), the estimator is confirmed to be efficient for this particular family of copulas.

However, and this may come as a surprise, in general, the pseudo-likelihood estimator is not efficient. The reason why this is so might even be more surprising: in the present setting, the empirical distribution functions are in general not efficient estimators of the marginal distribution functions.

But how can it be that, although we did not assume anything about the marginal distributions at all except for continuity, the empirical distribution functions are not efficient? The reason is that provided we would know what the true copula is, a kind of transfer of information from one margin to the other would be possible. This information is neglected by the collection of empirical distribution functions. In other words, the usual copula paradigm of separating the margins from the dependence involves a non-significant loss of information.

The following stylized example in dimension $d = 2$ helps to illustrate the phenomenon. Assume that the true copula corresponds to the uniform distribution on the union of the two squares $[0, 1/2] \times [1/2, 1]$ and $[1/2, 1] \times [0, 1/2]$. A scatterplot of a typical sample from a distribution with this particular copula will consist of two clouds of points, one cloud lying on the north-west of the other one. The median of the marginal distribution in the horizontal direction will then be located at the right of the left-hand cloud and at the left of the right-hand cloud; similarly for the median in the vertical direction. Hence, provided the marginal distributions have uniformly bounded densities, we will be able to locate the two medians up to an error of order $O(n^{-1})$, where $n$ denotes the sample size. This estimation accuracy is to be contrasted with the usual rate $O(n^{-1/2})$ available from the empirical quantile function.

In general, the amount of improvement that can be expected will not be as spectacular as in the above example. For precise conditions under which maximizing the pseudo-likelihood does yield an asymptotically semiparametrically efficient estimator of dependence parameters in copula models, see Genest and Werker (2002). Still, the examples on pages 295–296 of Bickel et al.