An Infinite-Dimensional Version of the Borsuk–Ulam Theorem*

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Received February 13, 2003

Abstract. We study the solvability of the equation $a(x) = f(x)$ on a sphere in a Banach space, where $a$ is a closed surjective linear operator and $f$ is an odd $a$-compact map. We also estimate the topological dimension of the solution set and give applications of the corresponding theorem to some problems in differential equations and other fields of mathematics.

Key words: closed surjective operator, compact map, operator equation.

The classical finite-dimensional Borsuk–Ulam theorem (e.g., see [1]) is well known. A generalization of this theorem to the case of infinite-dimensional Banach spaces was proved in [2].

Theorem 0. Let $E_1$ and $E_2$ be Banach spaces, let $a: E_1 \rightarrow E_2$ be a surjective continuous linear operator, let $S_r(0) \subset E_1$ be the sphere of radius $r$ centered at zero, and let $f: S_r(0) \rightarrow E_2$ be an odd compact map.

If $\dim(\ker a) \geq 1$, then the equation

$$a(x) = f(x) \quad (1)$$

has a nonempty solution set $N(a, f)$ and $\dim(N(a, f)) \geq \dim(\ker a) - 1$.

Here $\dim(N(a, f))$ is the topological dimension of $N(a, f)$.

Similar equations in which $a: D(a) \subset E_1 \rightarrow E_2$ is a closed surjective operator naturally arise in many problems in the theory of ordinary and partial differential equations (e.g., see [3, 4]). In this paper, we extend Theorem 0 to the case of closed surjective linear operators $a$ (not necessarily everywhere defined) perturbed by $a$-compact operators. Let us give some definitions.

Let $a: D(a) \subset E_1 \rightarrow E_2$ be a closed surjective linear operator.

Definition 1. We say that a continuous map $f: X \subset E_1 \rightarrow E_2$ is $a$-compact if the set $f(B \cap a^{-1}(A))$ is compact for arbitrary bounded sets $A \subset E_2$ and $B \subset X$. (The empty set is compact by definition.)

Let us derive a necessary and sufficient condition for the $a$-compactness of $f$.

The graph norm makes $D(a)$ a Banach space, which we denote by $E$. Clearly, the embedding $j: E \rightarrow E_1$ is Lipschitzian. Let $X \subset D(a)$. We set $\tilde{X} = j^{-1}(X)$ and consider the map $\tilde{f}: \tilde{X} \rightarrow E_2$ defined by the formula $\tilde{f}(x) = f(j(x))$.

Proposition 1. A continuous map $f$ is $a$-compact if and only if $\tilde{f}$ is compact.

The proof of Proposition 1 is not complicated, and we omit it.

Let a map $f: D(f) \subset S_r(0) \rightarrow E_2$ satisfy the following conditions:

1. $D(f) = D(a) \cap S_r(0)$;
2. $f(-x) = -f(x)$ for all $x \in D(f)$;
3. $f$ is $a$-compact.

The main result of this paper is the following theorem.

Theorem 1. Let $f$ satisfy conditions (1)–(3). If the kernel $\ker a$ is nontrivial, then Eq. (1) on the sphere $S_r(0)$ has a nonempty solution set $N(a, f)$ with $\dim(N(a, f)) \geq \dim(\ker a) - 1$. (The case $\dim(\ker a) = \infty$ is not excluded.)

Note that Theorem 1 implies Theorem 0, but the converse implication is not obvious. Indeed, if $a$ is a closed surjective operator and $f: D(a) \rightarrow E_2$ is odd and $a$-compact, then the equation

\*Supported by the Russian Foundation for Basic Research, Grant No. 02-01-00189.
\(a(j(x)) = f(j(x))\) has a solution on the sphere \(S_t\) in the space \(E\) by Theorem 0. Consequently, there exist solutions of Eq. (1) on the set \(S' = j(S_t)\), which, however, is not a sphere in \(E_1\).

1. Proof of the Borsuk–Ulam theorem. Let \(a: D(a) \subset E_1 \to E_2\) be a closed surjective linear operator, and let \(L = \text{Ker}(a)\). Let \(p\) be the projection of \(E_1\) onto the quotient space \(E = E_1/\text{Ker}(a)\). Consider the map \(a_1: D(a_1) \subset E \to E_2\), where \(D(a_1) = p(D(a))\) and \(a_1([x]) = a(x)\). It is well known (e.g., see [4]) that \(a_1\) is invertible and

\[
\|a_1^{-1}\| = \sup_{y \in E_2} \frac{\|a_1^{-1}(y)\|}{\|y\|} = \sup_{y \in E_2} \frac{\inf\{\|x\| : x \in E_1, a(x) = y\}}{\|y\|}.
\]

We set \(\|a_1^{-1}\| = \beta(a)\).

**Lemma 1.** The map \(a^{-1}\) is a Lipschitz set-valued map with Lipschitz constant \(\beta(a)\):

\[
h(a^{-1}(x_1), a^{-1}(x_2)) \leq \beta(a)\|x_1 - x_2\|,
\]

where \(h\) is the Hausdorff metric in \(E_1\).

**Lemma 2.** For each \(k > \beta(a)\), there exists an odd continuous map \(q: E_2 \to E_1\) such that

1. \(a(q(y)) = y\) for all \(y \in E_2\);
2. \(\|q(y)\| \leq k\|y\|\).

**Proof of Theorem 1.** First, we prove that \(N(a, f)\) is nonempty.

Consider the map \(g: D(a) \subset E_1 \to E_2\) defined by the formula

\[
g(x) = \begin{cases} \frac{\|x\|}{r} f\left(\frac{rx}{\|x\|}\right) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}
\]

Note that \(g\) is also \(a\)-compact. Indeed, let \(A\) be a bounded subset in \(E_2\), and let \(B \subset (B_R(0) \cap D(a))\).

Then

\[g(B \cap a^{-1}(A)) \subset C = \{tu : t \in [0, R/r], u \in f(S_t(0) \cap a^{-1}(A))\}.\]

Since \(C\) is relatively compact, it follows that \(g(B \cap a^{-1}(A))\) is compact.

Let \(q\) be a map satisfying the conditions in Lemma 2, let \(b \in \text{Ker}(a)\) and \(b \neq 0\), and let \(E = E_2 \times \mathbb{R}^1\). Consider the map \(\alpha: E \to E_2\) given by \(\alpha(y, t) = g(q(y) + tb)\). It is clearly odd. Let us verify that it is compact.

Let \(A = A_1 \times [a, c]\) be a bounded set in \(E\). Then \(A_2 = \{q(y) + tb : (y, t) \in A\}\) is also a bounded set, and \(A_2 \subset a^{-1}(A_1)\). By the \(a\)-compactness of \(g\), we conclude that the set \(g(A_2) = \alpha(A)\) is relatively compact.

Consider the equation

\[\alpha(y, t) = y\]

on the unit sphere \(S \subset E\). One can readily prove (e.g., see Theorem 0) that this equation has a solution, i.e., there exists a point \((y_0, t_0) \in S\) such that \(g(q(y_0) + t_0b) = y_0\). Note that \(z_0 = q(y_0) + t_0b \neq 0\).

It follows that

\[g(z_0) = \frac{\|z_0\|}{r} f\left(\frac{rz_0}{\|z_0\|}\right) = y_0,
\]

and consequently,

\[f\left(\frac{rz_0}{\|z_0\|}\right) = \frac{r}{\|z_0\|} y_0.
\]

On the other hand,

\[a(z_0) = a(q(y_0) + t_0b) = a(q(y_0)) + t_0a(b) = y_0.
\]

Therefore, \(a(rz_0/\|z_0\|) = (r/\|z_0\|)y_0\). Let \(x_0 = rz_0/\|z_0\| \in S_r(0) \subset E_1\). Then \(f(x_0) = a(x_0)\); i.e., \(x_0\) is a solution of Eq. (1), which proves the first part of the theorem.

The proof of the second part, where the dimension of \(N(a, f)\) is estimated, can be carried out by analogy with the corresponding argument in the proof of Theorem 0.