Nonremovable Zero Lyapunov Exponents*

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Abstract. Skew products over a Bernoulli shift with a circle fiber are studied. We prove that in the space of such products there exists a nonempty open set of mappings each of which possesses an invariant ergodic measure with one of the Lyapunov exponents equal to zero. The conjecture that the space of $C^2$-diffeomorphisms of the 3-dimensional torus into itself has a similar property is discussed.

Keywords: Lyapunov exponent, partially hyperbolic system, nonuniform hyperbolicity, dynamical system, skew product, Bernoulli diffeomorphism.

1. Introduction

To what extent is the behavior of a generic dynamical system hyperbolic?

A number of problems in modern theory of smooth dynamical systems can be viewed as some forms of this question. It was shown in the 1960s that uniformly hyperbolic systems (Anosov diffeomorphisms, Axiom A) are not dense in the space of dynamical systems [1]. This necessitated weakening the notion of hyperbolicity. The notions of partial and (important here) nonuniform hyperbolicity appeared (Pesin’s theory [2]). In Pesin’s theory, hyperbolic behavior is characterized by nonzero Lyapunov exponents for some invariant measure. The most natural case is that of a system with a smooth invariant measure. This case was studied in various aspects, for example, in [3–7]. However, the question about Lyapunov exponents can also be considered for maps that do not a priori carry a natural invariant measure. An invariant measure is said to be good if it can be obtained from the Lebesgue measure by the Krylov–Bogolyubov procedure.

Problem 1. Does a generic smooth dynamical system on a compact Riemannian manifold have nonzero Lyapunov exponents for each good measure?

This problem (in a slightly different form) was posed by M. Shub and A. Wilkinson [6] in connection with the question about the existence of SRB-measures for a generic dynamical system and is still open. Even if one removes any conditions on the invariant measures, the problem remains open and meaningful.

Conjecture 1. In the space of diffeomorphisms of the three-dimensional torus, there exists an open set of mappings having an ergodic invariant measure with one of the Lyapunov exponents equal to zero.

This conjecture suggests that the answer in Problem 1 may be negative, although the conjecture and the problem are apparently far from each other.

We intend to present the proof of the conjecture in a series of papers, where M. Nalsky is the main author. The present paper is the first work in the series.

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2. Main Results

The following theorem is the main result of the paper.

Consider the Bernoulli shift \( \sigma : \Sigma^2 \to \Sigma^2 \) on the space of two-way sequences of zeros and ones. Let

\[ f_j : S^1 \to S^1, \quad j = 0, 1, \]  

be two diffeomorphisms of a circle. We set \( M = \Sigma^2 \times S^1 \). Consider the step skew product

\[ F : M \to M, \quad (\omega, x) \mapsto ((\sigma \omega, f_{\omega_0}(x))). \]  

We use the term “step skew product” since the map on a fiber depends only on the zeroth element of the corresponding sequence in the base and hence resembles a step function on \( \Sigma^2 \).

**Theorem 1.** In the space \( (\text{Diff}^1(S^1))^2 \) of pairs of diffeomorphisms of the circle equipped with the \( C^1 \)-topology, there exists an open set \( U \) such that for each pair in \( U \) the corresponding step skew product (2) has an invariant ergodic measure with zero Lyapunov exponent along the fiber.

The definition of Lyapunov exponent along the fiber for the map (2) is given at the beginning of Sec. 4.

Theorem 1 follows from Theorems 2 and 3.

**Theorem 2.** Consider the set of pairs of mappings (1) such that

(i) the action of the semigroup \( G^+(f_0, f_1) \) generated by the diffeomorphisms \( f_0 \) and \( f_1 \) is minimal, that is, each orbit is dense in \( S^1 \);

(ii) for each point \( x \in S^1 \), there exists a \( j \in \{0, 1\} \) such that \( |f_j'(x)| > 1 \);

(iii) there exists a diffeomorphism \( f \in G^+(f_0, f_1) \) with a hyperbolic attracting periodic point.

Then the corresponding step skew product (2) has an ergodic invariant measure with zero Lyapunov exponent along the fiber.

**Remark 1.** It follows from an analysis of the proofs and some considerations like the compactness of the circle that one can weaken condition (ii) in the following way:

(ii') for each point \( x \in S^1 \), there exists an element \( g \in G^+(f_0, f_1) \) such that

\[ |g'(x)| > 1. \]

**Supplement to Theorem 2.** The measure in Theorem 2 can be taken to be nonatomic.

**Theorem 3.** The space \( (\text{Diff}^1(S^1))^2 \) contains a nonempty open subset of pairs possessing properties (i), (ii), and (iii).

We intend to derive the conjecture stated above from Theorem 1 in the following way.

Each system \( F \) of the form (2) in Theorem 1 admits a “smooth realization” in the sense of [10]. Namely, there exists a smooth self-map \( \mathcal{F} \) of the three-dimensional torus with an invariant partially hyperbolic set \( \Lambda \) foliated by circles such that the following holds. The restriction of \( \mathcal{F} \) to \( \Lambda \) is topologically conjugated to \( F \), and the restriction of the conjugacy to each central fiber is a smooth map. A small perturbation of \( \mathcal{F} \) has an invariant set homeomorphic to \( \Lambda \) (see [8]).

In a subsequent paper, which is now in preparation, we intend to prove that the conjecture holds for all maps in a small neighborhood of \( \mathcal{F} \). Note that for the complete realization of this program one should take an initial map \( \mathcal{F} \) of class \( C^2 \) and consider a \( C^2 \)-small neighborhood of \( \mathcal{F} \). In the present paper, we consider a \( C^1 \)-map \( F \) and a \( C^1 \)-neighborhood in the space of step skew products.

This plan essentially uses the approach in [9–11] and is a continuation of these studies.

3. The Structure of the Paper

To prove Theorem 2, we construct the desired measure as a limit of invariant measures that are uniformly distributed on periodic orbits and whose Lyapunov exponents tend to zero. Moreover, these periodic orbits will be in some sense “similar” to each other.

The idea is realized as follows. In Sec. 4, we show that if the limit of a sequence of ergodic measures is ergodic, then the Lyapunov exponent along the fiber for the limit measure is the