Projective Characters of the Infinite Generalized Symmetric Group

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Abstract. We consider the infinite generalized symmetric group $B_\infty^\infty \times \mathbb{Z}_m^\infty$, introduce its covering $\tilde{B}_m$, and describe all indecomposable characters on the group $\tilde{B}_m$.

Key words: projective representation, infinite generalized symmetric group, indecomposable character, covering group.

By definition, the generalized symmetric group $B_\infty^{\infty}$ is the semidirect product $S(\infty) \ltimes \mathbb{Z}_m^{\infty}$, where $S(\infty)$ is the group of all finite permutations of $\mathbb{N}$ and $\mathbb{Z}_m^{\infty}$ is the set of sequences $\{z_j\}_{j=1}^{\infty}$, with $\bar{z}_j = 0$ for all but finitely many $j$, so that $S(\infty)$ naturally acts on $\mathbb{Z}_m^{\infty}$. Here we announce a complete classification of finite type projective factor representations of the group $B_\infty^{\infty}$. To these representations, the well-known construction in [4] assigns ordinary factor representations of the covering group

$$\tilde{B}_m = \langle \tilde{s}_i, \tilde{w}_j \mid i, j \in \mathbb{N} \rangle \mid \tilde{s}_i^2 = 1; \tilde{w}_i^m = 1; (\tilde{s}_i \tilde{s}_{i+1})^3 = 1; \tilde{s}_i \tilde{w}_i = \tilde{w}_{i+1} \tilde{s}_i;$$

$$\tilde{s}_i \tilde{w}_j = \vartheta \tilde{w}_j \tilde{s}_i, j \neq i, i + 1; (\tilde{s}_i \tilde{s}_j)^2 = \nu, |i - j| > 1; \tilde{w}_i \tilde{w}_j = \mu \tilde{w}_j \tilde{w}_i, i \neq j),$$

(1)

where $\vartheta$, $\nu$, and $\mu$ are central elements such that $\vartheta^{(2,m)} = \nu^2 = \mu^{(2,m)} = 1$. (Here $(2, m)$ is the least common divisor of 2 and $m$.) A similar problem for the group $S(\infty)$ was solved by Nazarov [5] (see also [3]). Clearly, the operators $\pi(\vartheta)$, $\pi(\nu)$, and $\pi(\mu)$ can differ only in sign from the identity operator for any unitary factor representation $\pi$ of the group $\tilde{B}_m^{\infty}$. On the other hand, the ordered triple $c(\pi)$ can be viewed as an element of the second cohomology group

$$\mathcal{H}^2(B_m^{\infty}, T) = \begin{cases} \mathbb{Z}_2^3 & \text{if } m = 2k \ (k \in \mathbb{N}), \\ \mathbb{Z}_2^2 & \text{if } m = 2k - 1 \end{cases}$$

(where $T$ stands for the one-dimensional torus) and hence defines some central extension of $B_m^{\infty}$. Clearly, $c(\pi \otimes \pi') = c(\pi)c(\pi')$ (pointwise multiplication).

1. Ordinary representations of $B_m^{\infty}$ were independently described in [1] and [2]. To compute the character of an ordinary representation, let us first parametrize conjugacy classes. For any $g \in B_m^{\infty}$, there exist $s \in S(\infty)$ and $w = (w_i^d) \in \mathbb{Z}_m^{\infty}$ such that $g = sw$. Let $\mathbb{N}/s$ be the set of orbits of the permutation $s$, and let $s(p)$ be the cycle that coincides with $s$ on the orbit $p$. For $p \in \mathbb{N}/s$, set $w(p) = \prod_{i \in p} w_i^d$. The unordered set of pairs $\{(|p|, w(p))\}_{p \in \mathbb{N}/s}$, where $|p|$ is the cardinality of an orbit $p$, is a complete conjugacy invariant in the group $B_m^{\infty}$.

Theorem 1. Let $\pi$ be a finite type factor representation of the group $B_m^{\infty}$. Then there exist two nonincreasing sequences $\alpha = \{\alpha_i\}$ and $\beta = \{\beta_i\}$ of positive numbers with $\sum (\alpha_j + \beta_j) < 1$, two sequences $\tilde{\alpha} = \{\tilde{\alpha}_i\}$ and $\tilde{\beta} = \{\tilde{\beta}_i\}$ of characters of the group $\mathbb{Z}_m$, and a normalized positive definite function $\zeta$ on $\mathbb{Z}_m$ such that the values of the character $\chi^{\tilde{\alpha} \tilde{\beta}}_{\beta \beta}$ of the representation $\pi$ are given by the formula

$$\chi^{\tilde{\alpha} \tilde{\beta}}_{\beta \beta}(sw) = \prod_{p \in \mathbb{N}/s} \left\{ \sum_{j} \alpha_j^{|p|} \tilde{\alpha}_i(w(p)) + (-1)^{|p| - 1} \beta_j^{|p|} \tilde{\beta}_i(w(p)) \right\} \zeta \zeta(w(p)), \quad (2)$$

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where

$$\delta_p = \begin{cases} 1 - \sum (\alpha_j + \beta_j) & \text{for } |p| = 1, \\ 0 & \text{for } |p| > 1. \end{cases}$$

2. Basic representations of the group $B^\infty_m$. Let $\mathcal{E} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathcal{I} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\mathcal{J} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$, and $\mathcal{K} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ be elements of the algebra $M_2(\mathbb{C})$ of complex $2 \times 2$ matrices, and let $\text{tr}_2$ be the normalized trace on $M_2(\mathbb{C})$. We define a representation $F$ of $\widetilde{B}^\infty_m$ on the generators by setting $F(\tilde{s}_k) = \mathcal{I}$ and $F(\tilde{w}_k) = (-1)^k \mathcal{J}$. Since $\mathcal{I} \mathcal{J} = -\mathcal{J} \mathcal{I}$, it follows that $c(F) = (I, -I, I)$. Next, let $B = \bigotimes_{i \in \mathbb{N}} M_i$ and $\tau = \bigotimes_{i \in \mathbb{N}} \tau_i$ ($M_i = M_2(\mathbb{C})$ and $\tau_i = \text{tr}_2$), $\mathcal{M}_{2n-1} = \mathcal{K}^{\otimes (n-1)} \otimes \mathcal{I} \otimes \mathcal{E}^\otimes \infty$, and $\mathcal{M}_{2n} = \mathcal{K}^{\otimes (n-1)} \otimes \mathcal{J} \otimes \mathcal{E}^\otimes \infty$. We treat $B$ as a $*$-subalgebra of the $II_1$-factor $M$ corresponding to the GNS-representation of $B$ for the state $\tau$. Define mappings $\Pi, \Phi: \widetilde{B}^\infty_m \to B$ on the generators by the relations

$$\Pi(\tilde{s}_k) = \frac{1}{\sqrt{2}} (\mathcal{M}_k - \mathcal{M}_{k+1}), \quad \Pi(\tilde{w}_k) = (-1)^k \mathcal{M}_k, \quad \Phi(\tilde{s}_k) = \frac{1}{\sqrt{2k}} (\sqrt{k-1} \mathcal{M}_{k-1} - \sqrt{k+1} \mathcal{M}_k) \quad (\text{see [5]}), \quad \Phi(\tilde{w}_k) = \mathcal{E}^\otimes \infty. \quad (3)$$

A straightforward argument shows that $\Pi$ and $\Phi$ extend to be $II_1$-factor representations of $\widetilde{B}^\infty_m$ such that $c(\Pi) = (-I, -I, I)$ and $c(\Phi) = (-I, I, I)$. For odd $m$, in view of the conditions imposed on the central elements $\vartheta, \nu, \mu$ in (1), only the representation $\Phi$ survives. Any other nontrivial triple can be obtained as the product of at most three distinct factors from the set $\{c(\Pi), c(\Phi), c(F)\}$. The corresponding characters of the tensor products of the representations $\Pi, \Phi$, and $F$, as well as the associated GNS-representations, are said to be basic.

**Theorem 2.** All basic representations are factor representations.

3. Classification of characters of the group $B^\infty_m$. The group $B^\infty_m$ is generated by the elements $s_i$ and $w_i$ satisfying relations (1) with $\vartheta = \nu = \mu = 1$. We define a homomorphism $\text{pr}: \widetilde{B}^\infty_m \to B^\infty_m$ on the generators by setting $\text{pr}(v) = 1$ for $v = \vartheta, \nu, \mu$, $\text{pr}(\tilde{s}_i) = s_i$, and $\text{pr}(\tilde{w}_i) = w_i$. If $b \in \widetilde{B}^\infty_m$ and $\text{pr}(b) = sw$, then $b = \prod_{p \in \mathbb{N}/s} b_p$, where $b_p \in \text{pr}^{-1}(s^{(p)}w(p))$.

**Theorem 3.** Let $\pi$ be a finite type factor representation of the group $\widetilde{B}^\infty_m$, let $\chi_\pi$ be the corresponding trace, and let $\pi_\mu$ be the basic representation such that $c(\pi_\mu) = c(\pi)$. Then there exist $\alpha, \beta, \tilde{\alpha},$ and $\tilde{\beta}$ (see Theorem 1) such that

$$\chi_\pi(b) = \chi_{\pi_\alpha}(b) \chi_{\beta \tilde{\beta}}^{\tilde{\alpha}}(\text{pr}(b)), \quad \text{where } b \in \widetilde{B}^\infty_m. \quad (5)$$

Note that, for given $\chi_\pi$ and $\chi_{\pi_\mu}$, the character $\chi_{\beta \tilde{\beta}}^{\tilde{\alpha}}$ is not uniquely determined from (5). This ambiguity is related to the properties of the zero set of the function $\chi_\pi$ and can be described explicitly for each extension.

**Sketch of Proof of Theorem 3.** The characters of factor representations are said to be indecomposable. One can prove the indecomposability of the character $\chi_\pi$ defined in (5) by using Theorem 2 and a multiplicativity argument.

As to the converse statement, we omit the technicalities and explain the main idea of the proof for the triplets $(-I, I, I)$ and $(-I, -I, I)$.

If $\pi$ is a factor representation of $\widetilde{B}^\infty_m$ and $c(\pi) = (-I, I, I)$, then, using the properties of conjugacy classes, one can prove that

$$\chi_\pi(b) = 0 \text{ if the permutation } s \text{ in the formula } \text{pr}(b) = sw \text{ has a cycle of even length}. \quad (6)$$