Moduli Spaces $\mathcal{M}_{2,1}$ and $\mathcal{M}_{3,1}$

Yu. Yu. Kochetkov*

Received December 15, 2008

Abstract. The cell structure of the spaces $\mathcal{M}_{2,1}$ and $\mathcal{M}_{3,1}$ is considered. These are the spaces of complex curves of genus 2 and 3 with one marked point. For the space $\mathcal{M}_{2,1}$, nine cells of the highest dimension 8 are described and their adjacency is studied. For the space $\mathcal{M}_{3,1}$, a list of all 1726 cells of the highest dimension 14 (with orientation) is obtained. The list of adjacent couples of cells is also obtained. These lists can be found on the web.

Key words: moduli space of curves with marked points, embedded graphs.

Introduction

By $\mathcal{M}_{g,1}$ we denote the moduli space of complex curves of genus $g$ with one marked point. This is a complex manifold of dimension $3g - 2$. Let $\Gamma$ be a graph embedded in a topological compact surface $S_g$ of genus $g$ in such a way that its complement is a disk. The Jenkins–Strebel construction allows uniquely defining a complex structure on $S_g$ by using the embedded graph $\Gamma$ and the set of length of its edges (see [2]). In what follows, we assume that the sum of edge lengths of the embedded graph is 1. It is known (e.g., see [1, Chap. 4]) that the space $\mathcal{M}_{g,1}$ has the following combinatorial description. Consider the set $\{\Gamma_1, \ldots, \Gamma_s\}$ of pairwise nonisomorphic trivalent graphs (each vertex of such a graph has valency 3) embedded in $S_g$ whose complements are homeomorphic to the 2-disk. Isomorphism here means isomorphism of embedded graphs; i.e., an isomorphism must preserve the cyclic order of edges when going counterclockwise around each vertex. Such a graph has $4g - 2$ vertices and $6g - 3$ edges.

Let us define a simplicial complex $\mathcal{M}_{g,1}^{\text{comb}}$. Its simplices $\Delta_1, \ldots, \Delta_s$ of the highest dimension $6g - 4$ are in one-to-one correspondence with the graphs $\Gamma_1, \ldots, \Gamma_s$. Each simplex $\Delta_i$ is isometric to the set

$$\{x_1 + \cdots + x_{6g-3} = 1, x_1, \ldots, x_{6g-3} > 0\}$$

in the space $\mathbb{R}^{6g-3}$. The numbers $x_1, \ldots, x_{6g-3}$ are the lengths of edges of the graph $\Gamma_i$. Note that the graph $\Gamma_i$ has no loops. Indeed, let $e$ be a loop with vertex $v$. As $\Gamma_i$ is a trivalent graph, it follows that three arcs go from $v$; two of them belong to the edge $e$ and the third defines the direction $\vec{n}$ of a normal vector to $e$ at $v$. The translation of $e$ in the direction opposite to $\vec{n}$ produces a noncontractible cycle that belongs to the complement of $\Gamma_i$. Therefore the contraction of the $j$th edge of $\Gamma_i$ gives us an embedded graph $\Gamma$ with $6g - 4$ edges and $4g - 3$ vertices whose complement is a disk. If the lengths $x_1, \ldots, x_{6g-3}$ of the edges of the graph $\Gamma_i$ are given, then we define the edge lengths $y_1, \ldots, y_{6g-4}$ of the graph $\Gamma$ by setting $y_k = x_k/x$ if $k < j$ and $y_k = x_{k+1}/x$ if $k \geq j$ (here $x = x_1 + \cdots + x_{j-1} + x_{j+1} + \cdots + x_{6g-3}$). Thus, $\Gamma$ defines a complex structure on $S_g$. With $\Gamma$ we associate a $(6g - 5)$-dimensional simplex $\Delta$ isometric to the standard simplex

$$\{y_1 + \cdots + y_{6g-4} = 1, y_1, \ldots, y_{6g-4} > 0\}$$

in the space $\mathbb{R}^{6g-4}$. We treat the simplex $\Delta$ as the $j$th face of the simplex $\Delta_i$.

The contractions of some edge of the graph $\Gamma_{i_1}$ and of some edge of the graph $\Gamma_{i_2}$ may produce isomorphic (embedded) graphs $\Gamma'$ and $\Gamma''$. In this case, we assume that the simplices $\Delta_{i_1}$ and $\Delta_{i_2}$ are adjacent in the complex $\mathcal{M}_{g,1}^{\text{comb}}$: they have the common $(6g - 5)$-dimensional face $\Delta' = \Delta''$.

*The work is supported by the Russian Foundation for Basic Research, grant 04-01-00647.
Each simplex $\Delta_i$ defines a $(6g - 4)$-dimensional cell in the moduli space $\mathcal{M}_{g,1}$, and the adjacency of simplices defines the adjacency of cells.

The graph $\Gamma_i$ may have a nontrivial automorphism group. Similarly, the graph obtained from $\Gamma_i$ by the contraction of an edge may have a nontrivial automorphism group as well. These automorphism groups define automorphism groups of the corresponding simplices of the complex $\mathcal{M}_{g,1}^{\text{comb}}$. After factorization by the actions of these groups, we obtain a topological space homeomorphic to the union of $(6g - 4)$- and $(6g - 5)$-dimensional cells in the moduli space $\mathcal{M}_{g,1}$.

Our aim is to describe the cell structure of the space $\mathcal{M}_{2,1}$ in dimensions 8 and 7 and the space $\mathcal{M}_{3,1}$ in dimensions 14 and 13. Note that the Deligne–Mumford compactification is not studied here.

1. The Space $\mathcal{M}_{2,1}$

The main result of this section is the following theorem.

**Theorem 1.** The space $\mathcal{M}_{2,1}$ consists of nine cells of the highest dimension 8. Their orientation and adjacency structure are described in Sections 1.2 and 1.3, in Table 1, and in Fig. 2.