Power Asymptotics of Spectral Functions of Boundary Value Problems for Generalized Second-Order Differential Equations with Boundary Conditions at a Singular Endpoint

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Abstract. Let \( I = (\infty, b) \), where \( b \leq +\infty \), and let \( M(x) \), \( x \in I \), be a nondecreasing function on \( I \) such that \( M(x) > 0 \) for \( x \in I \). In the middle of the past century, it was proved that, in the case where \( M(x) \) is Lebesgue integrable on the interval \( (\infty, c) \), \( c \in I \), the boundary value problem

\[
-\frac{y''(x)}{M(x)} = \lambda y(x), \quad x \in I, \quad \lim_{x \to \infty} y(x) = 1
\]

is uniquely solvable for any complex \( \lambda \) and has at least one spectral function \( \tau(\lambda) \) (\( +^{r} \) denotes right derivative).

A result relating the asymptotic behavior of \( M(x) \) as \( x \to -\infty \) to that of \( \tau(\lambda) \) as \( \lambda \to +\infty \) is announced. Similar results are also announced for two other boundary value problems with boundary conditions at a singular endpoint.

Key words: string, boundary value problem, singular endpoint, spectral function.

1. A generalized second-order linear differential equation. By \( \mathcal{I} \) we denote the interval of the real line with endpoints \( a \) and \( b \) \((\infty \leq a < b \leq +\infty)\). Let \( \mathcal{M}(x) \), \( x \in \mathcal{I} \), be a nondecreasing function, which may have intervals of constancy, an absolutely continuous component, a continuous singular component, and points of discontinuity; moreover, the set of such points may be dense in \( \mathcal{I} \). We denote the set of growth points of \( \mathcal{M}(x) \) by \( \mathcal{I}_{\mathcal{M}} \). Let \( \inf_{x \in \mathcal{I}} \mathcal{M}(x) = a_{0}(\geq a) \), and let \( \sup_{x \in \mathcal{I}} \mathcal{M}(x) = b_{0}(\leq b) \). We fix a function \( \mathcal{Q}(x) \), \( x \in \mathcal{I} \), being the difference of two nondecreasing functions. We say that the left (right) endpoint of the interval \( \mathcal{I} \) is regular if \( a_{0} > -\infty \) (respectively, \( b_{0} < +\infty \)), \( \inf_{x \in \mathcal{I}} \mathcal{M}(x) > -\infty \) (respectively, \( \sup_{x \in \mathcal{I}} \mathcal{M}(x) < +\infty \)), and \( \mathcal{Q}(x) \) is of bounded variation in a right (left) half-neighborhood of \( a_{0} \) (respectively, of \( b_{0} \)). Otherwise, this endpoint is said to be singular.

As early as in [1], a generalized differential operation \( l_{\mathcal{I}, \mathcal{M}}[\_] \) was introduced; its action on a function \( f \) is roughly expressed by

\[
l_{\mathcal{I}, \mathcal{M}}[f](x) = -\frac{d}{d\mathcal{M}(x)} \left( f^{+}(x) - \int_{c+0}^{x+0} f(s) d\mathcal{Q}(x) \right),
\]

where \( c \) is a fixed point in \( \mathcal{I} \) and \( f^{+}(x) \) denotes the right derivative of \( f \) at \( x \). The same paper [1] also considered the generalized second-order differential equation

\[
l_{\mathcal{I}, \mathcal{M}}[y] - \lambda y = 0,
\]

in which \( \lambda \) is a complex parameter.

We regard a function \( u(x) \) as a solution Eq. (1) if it is absolutely continuous on \( \mathcal{I} \) and has a right derivative \( u^{+}(x) \) at each point \( x \in \mathcal{I} \), the function \( u^{+}(x) - \int_{c+0}^{x+0} u(s) d\mathcal{Q}(s) \) is absolutely continuous with respect to the measure generated by \( \mathcal{M}(x) \) (or, briefly, \( \mathcal{M} \)-absolutely continuous), and the relation \( l_{\mathcal{I}, \mathcal{M}}[u](x) - \lambda u(x) = 0 \) holds \( \mathcal{M} \)-almost everywhere in \( \mathcal{I} \).

In this paper, we consider only the case where the left endpoint is singular. Thus, we assume that \( a \notin \mathcal{I} \). For simplicity of exposition, we also assume that \( b \notin \mathcal{I} \); otherwise, we would have to introduce the right adjoint derivative at \( x = b \) (see [2] and [3]). This is also justified by that Theorems A, B, C, and D stated below are valid independently of the behavior of the functions \( \mathcal{M}(x) \) and \( \mathcal{Q}(x) \) at the right endpoint of the interval \( \mathcal{I} \).
Note that the weighted Sturm–Liouville equation $-y'' + q(x)y = \lambda \rho(x)y$, $a < x < b$, in which $\rho(x) \geq 0$ and $q(x)$ are real measurable locally integrable functions, is a special case of Eq. (1). This is the case in which $\mathcal{M}(x)$ and $\mathcal{D}(x)$ are locally absolutely continuous and, moreover, $\rho(x) = \mathcal{M}'(x)$ and $q(x) = \mathcal{D}'(x)$ almost everywhere in $(a,b)$ with respect to the Lebesgue measure. When $\mathcal{D}(x) = \text{const}$, Eq. (1) is the so-called string equation for a string $S(\mathcal{I}, \mathcal{M})$ stretched by a unit force along the interval $\mathcal{I} = (a,b)$, for which $\mathcal{M}(x)$ is the mass distribution function (see [2] and [3]).

2. Spectral functions. We use the definition of a spectral function given below. Let $L^2(\mathcal{I})$ be the quasi-Hilbert space of all complex-valued $\mathcal{M}$-measurable functions $f(x)$ on $\mathcal{I}$ satisfying the condition \(||f||^2_\mathcal{M} := \int_{\mathcal{I}} |f(x)|^2 \, d\mathcal{M}(x) < \infty\). We include a function $f(x)$ from $L^2(\mathcal{I})$ in a set $\hat{L}^2(\mathcal{I})$ if each singular endpoint of the interval $\mathcal{I}$ has a half-neighborhood contained in $\mathcal{I}$ in which $f(x)$ identically vanishes. If both endpoints are regular, then $\hat{L}^2(\mathcal{I}) = L^2(\mathcal{I})$.

Let $G$ be a fixed set of functions defined on $\mathcal{I}$ such that, for each $\lambda \in \mathbb{R}$, Eq. (1) has precisely one solution belonging to $G$. We denote the problem of finding such a solution by $\mathcal{G}$. Generally speaking, it can be written in the form $l_{\mathcal{I}, \mathcal{M}}[y] = \lambda y$, $y \in G$. However, the condition $y \in G$ is usually replaced by a condition determining the membership of functions in $G$.

**Definition.** We say that a nondecreasing function $\tau(\lambda)$, $-\infty < \lambda < +\infty$, normalized by the conditions $\tau(\lambda) = \frac{1}{2}(\tau(-\lambda) + \tau(\lambda))$ for any $\lambda \in \mathbb{R}$ and $\tau(0) = 0$ is a spectral function (or, briefly, a SF) of the problem $\mathcal{G}$ if the transformation $\Upsilon$: $f \mapsto \mathcal{F}$ defined by $\mathcal{F}(\lambda) = \int_{\mathcal{I}} u(x, \lambda)f(x) \, d\mathcal{M}(x)$, where $u(x, \lambda)$ is a solution of this problem, is an isometry of $\hat{L}^2(\mathcal{I})$ to $L^2(\tau(-\infty, +\infty))$, i.e., for any function $f \in \hat{L}^2(\mathcal{I})$ and its image $\mathcal{F}(\lambda) = (\Upsilon f)(\lambda)$, the Parseval identity $\int_{\mathcal{I}} |f(x)|^2 \, d\mathcal{M}(x) = \int_{-\infty}^{+\infty} |\mathcal{F}(\lambda)|^2 \, d\tau(\lambda)$ holds. A spectral function $\tau(\lambda)$ is said to be orthogonal if $\Upsilon$ takes $\hat{L}^2(\mathcal{I})$ to a dense subset of the space $L^2(-\infty, +\infty)$. The set of growth points of a SF $\tau(\lambda)$ is called the spectrum of this SF.

3. Three boundary value problems for the differential equation (1) with boundary condition at a singular endpoint.

3.A. Strings of class $\mathcal{M}_1$. We include a string $S(\mathcal{I}, \mathcal{M})$ in a class $\mathcal{M}_1$ if $\mathcal{I} = (-\infty, b)$, where $b \leq +\infty$, $\inf \mathcal{I} = -\infty$, and the function $\mathcal{M}(x)$ is Lebesgue integrable on the interval $(-\infty, c)$ for some $c \in \mathcal{I}$ (the last condition can hold only when $\mathcal{M}(x) = o(|x|^{-1})$ as $x \to -\infty$).

With each string $S(\mathcal{I}, \mathcal{M}) \in \mathcal{M}_1$ we associate the boundary value problem

$$-\frac{d}{d\mathcal{M}(x)} y^+(x) = \lambda y(x), \quad x \in \mathcal{I}, \quad \lim_{x \downarrow -\infty} y(x) = 1;$$

we refer to a spectral function of this problem as a spectral function of this string.

3.B. Strings of class $\mathcal{M}_0$. We include a string $S(\mathcal{I}, \mathcal{M})$ in a class $\mathcal{M}_0$ if $\mathcal{I} = (0, b)$, where $0 < b \leq +\infty$, and $\lim_{x \to 0} \mathcal{M}(x) = -\infty$ but the function defined by $f(x) = x$ for any $x \in \mathcal{I}$ is integrable with respect to the measure generated by $\mathcal{M}$ on the interval $(0, c)$ for some $c \in \mathcal{I}$. With each string $S(\mathcal{I}, \mathcal{M}) \in \mathcal{M}_0$ we associate the boundary value problem

$$-\frac{d}{d\mathcal{M}(x)} y^+(x) = \lambda y(x), \quad x \in \mathcal{I}, \quad \lim_{x \downarrow 0} \frac{y(x)}{x} = 1;$$

we refer to a spectral function of this problem as a spectral function of this string.

3.C. Boundary value problem for a generalization of the “standard” boundary value problem

$$-y'' + \frac{\nu^2 - 1/4}{x^2} y = \lambda y, \quad 0 < x < +\infty, \quad \lim_{x \downarrow 0} \frac{y(x)}{x^{\nu+1/2}} = 1.$$

Here we consider the boundary value problem

$$-y'' + \left(\frac{\nu^2 - 1/4}{x^2} + q_1(x)\right) y = \lambda \rho(x)y, \quad 0 < x < c, \quad \lim_{x \downarrow 0} \frac{y(x)}{x^{\nu+1/2}} = 1,$$