A note on solution sets of interval-valued fuzzy relational equations

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1 Introduction

Fuzzy relational equations were first investigated by Sanchez (1976, 1977). They play an important role in the applications of fuzzy sets and systems. A system of fuzzy relational equations with max-$T$ composition is of the form

$$\max_{1 \leq i \leq m} T(x_i, a_{ij}) = b_j, \quad j = 1, 2, \cdots, n,$$

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where \(a_{ij}, b_j, \text{ and } x_i\) are real numbers in the unit interval \([0, 1]\) and \(T : [0, 1]^2 \rightarrow [0, 1]\) is a triangular norm (\(t\)-norm for short), i.e., a binary operator such that \(T\) is commutative, associative, monotone in each argument and satisfies the boundary condition \(T(x, 1) = x\) for each \(x \in [0, 1]\). Fuzzy relational equations in the above form are also called max-\(T\) equations and denoted as \(x \circ_T A = b\) in the matrix form where “\(\circ_T\)” denotes the max-\(T\) composition, \(A = (a_{ij})_{m \times n}\) is a coefficient matrix, \(b = (b_1, b_2, \ldots, b_n)\) is a right-hand side vector and \(x = (x_1, x_2, \ldots, x_m)\) is an unknown vector. Typically, the involved \(t\)-norm \(T\) is required to be continuous, i.e., continuous as a function of two arguments. The most frequently used continuous \(t\)-norm is the minimum operator \(T_M(x, y) = \min(x, y)\). Other important continuous \(t\)-norms include the product operator \(T_P(x, y) = xy\) and Łukasiewicz \(t\)-norm \(T_L = \max(x + y - 1, 0)\). For more details on triangular norms, readers may refer to the monograph of Klement et al. (2000).

For a system of max-\(T\) equations \(x \circ_T A = b\), it is well known that its consistency can be verified by constructing and checking a potential maximum solution in polynomial time. The set of all solutions, when it is non-empty, is a finitely generated root system which can be fully determined by a unique maximum solution and a finite number of minimal solutions. However, the detection of all minimal solutions is an NP-hard problem. Readers may refer to Li and Fang (2008) and references therein for a detailed discussion on max-\(T\) equations. Note that similar conclusions can be drawn for a system of max-\(T\) inequalities.

It has been observed for a long time that the consistency of a system of max-\(T\) equations is very sensitive to the data. The inaccurate or deficient data may lead to an inconsistent system of max-\(T\) equations when modeling a system via fuzzy relations. To deal with the uncertainty of the data and resolve the inconsistency of the system, one possible method is to consider the interval-valued max-\(T\) equations, i.e., each entry in the matrix \(A\) and the vector \(b\) is replaced by a closed interval of possible values in \([0, 1]\). A system of interval-valued max-\(T\) equations can be represented in the form \(x \circ_T \tilde{A} = \tilde{b}\) where \(\tilde{A} = (\tilde{a}_{ij})_{m \times n}\) is an interval-valued matrix with \(\tilde{a}_{ij} = [\bar{a}_{ij}, \underline{a}_{ij}]\) and \(\tilde{b} = (\underline{b}_j)_{1 \times n}\) is an interval-valued vector with \(\underline{b}_j = [\underline{b}_j, \bar{b}_j]\). Denote \(\bar{A} = (\bar{a}_{ij})_{m \times n}, \underline{A} = (\underline{a}_{ij})_{m \times n}\) and similarly, \(\bar{b} = (\bar{b}_j)_{1 \times n}, \underline{b} = (\underline{b}_j)_{1 \times n}\). By extending the natural order in a componentwise manner, \(\tilde{A}\) and \(\tilde{b}\) can be denoted as \(\tilde{A} = [\bar{A}, \underline{A}] = \{A = (a_{ij})_{m \times n} | \bar{A} \leq A \leq \underline{A}\}\) and \(\tilde{b} = [\bar{b}, \underline{b}] = \{b = (b_j)_{1 \times n} | \bar{b} \leq b \leq \underline{b}\}\), respectively. Without loss of generality, we may always assume that \(\underline{A} \leq A\) and \(\bar{b} \leq b\). In Wang et al. (2003), three types of solutions of a system of interval-valued max-\(T\) equations were discussed in detail. However, as pointed out by Wang et al. (2007), a critical lemma presented in these claims is not correct and consequently some statements are deficient although the major claims remain valid.

In this note, we prove the major results in Wang et al. (2003) in a succinct manner. Some slightly different notations are used for readability. Basic results of fuzzy relational equations and fuzzy relational inequalities are summarized in Sect. 2. The relationship between interval-valued fuzzy relational equations and fuzzy relational inequalities is illustrated in Sect. 3.