FAST SOLUTION OF 3D-ELASTICITY PROBLEM OF A PLANAR CRACK OF ARBITRARY SHAPE

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Abstract. A planar crack of an arbitrary shape in a homogeneous elastic medium is considered. The problem is reduced to integral equation for the crack opening vector. Its numerical solution utilizes Gaussian approximating functions that drastically simplify construction of the matrix of a linear algebraic system of the discretized problem. For regular grids of approximating nodes, this matrix turns out to have the Toeplitz structure. It allows one to use the Fast Fourier Transform algorithms for calculation of the matrix-vector products in the process of iterative solution of the discretized problem. The method is applied to a crack bounded by the curve for for . The contribution of a crack to the overall effective elastic constants is calculated.

Keywords: Crack, Fast Fourier Transform, Effective elastic properties.

1. Introduction. The problem of elasticity for a medium with a planar crack of an arbitrary shape is of obvious relevance for applications. Usually, problems of this kind are reduced to an integral equation for the crack opening vector, and numerical methods are applied to its solution. If conventional approximation functions (e.g., polynomial splines) are used for the discretization, the crack area should be divided into boundary elements, and a great portion of the computer time is spent on construction of the matrix of the discretized system that involves numerical integration of a singular kernel over the boundary elements.

In the present work, a class of Gaussian approximating functions is used for the numerical solution of the integral equation of the crack problem. This has several important advantages. Firstly, the method based on these functions is mesh free, and only the coordinates of the nodes of the approximation need to be known for the construction of the matrix of discretized problem. Secondly, for the Gaussian functions, the elements of this matrix may be found in explicit analytic forms and calculated very fast. An additional advantage holds for regular node grids. In this case, the matrix of the discretized system has Toeplitz’s structure, and the matrix-vector product operation may be carried out by the Fast Fourier Transform algorithms. As a result, the time of the construction of the iterative solution of the discretized system is reduced very substantially.

We consider a planar crack of arbitrary shape. The convergence of the method is illustrated on the circular crack. A specific class of cracks is considered, and their contributions to the effective elastic constants of the cracked medium are found.
2. Integral equation of the crack problem. Let an arbitrary crack occupy a smooth surface \( \Omega \) in a homogeneous isotropic medium, and be subjected to an external stress field \( \sigma^0(x) \). The crack is supposed to be opened by the field \( \sigma^0(x) \), and the vector of the displacement jump on the crack surface is \( b(x) \), \( x \) is a point of 3D-space. It is known (see Kanaun (1981), Kanaun and Levin (2008)) that the vector \( b(x) \) satisfies the following integral equation

\[
\int_\Omega T_g(x,x') b_j'(x') d\Omega = \sigma^0_j(x)m_j(x), \quad x \in \Omega,
\]

\[
T_{ij}(x,x') = -m_i(x)S_{ijkl}(x-x')m_l(x'),
\]

\[
S_{ijkl}(x) = C_{ijmn}K_{mpq}(x)C_{pkl} - C_{ijkl} \delta(x), \quad K_{ijkl}(x) = \left[ \partial_i \partial_j G_{jk}(x) \right]_{ji(hl)}.
\]

Here \( m(x) \) is the normal to the crack surface \( \Omega \), \( G(x) \) is the Green tensor of the medium, \( \delta(x) \) is 3D-Dirac’s delta-function. The integral in eq (1) formally diverges because \( T(x,x') \propto (x-x')^3 \), when \( x \to x' \). The regularization of this integral was indicated in Kanaun (1981), and if \( \Omega \) is a plane surface, \( m \) is a constant vector, \( T(x,x') = T(x-x') \), and this integral should be understood in the following sense

\[
\int_\Omega T(x-x')b(x')d\Omega = p.v.\int_\Omega T(x-x')[b(x')-b(x)]dP'.
\]

Here the integration is performed over the entire plane \( P \) that includes the surface \( \Omega \), \( p.v. \) is the Cauchy principal value of the integral, the function \( b(x) \) is continued by zero outside \( \Omega \). Note that if a function \( \varphi(x) \) and its Fourier transform \( \varphi^*(k) \) tend to zero at infinity faster than any negative power of \( |x| / |k| \), the action of the operator \( T \) on this function may be defined by the equation

\[
(T\varphi)(x) = \frac{1}{(2\pi)^3} \int T^*(k)\varphi^*(k) \exp(-ik \cdot x) dk.
\]

Here the Fourier transform \( T^*(k) \) of the generalized function \( T(x) \) has the form

\[
T^*_i(k) = \mu \left| k \right| \left[ \delta_{ij} + \eta (m_i m_j + e_i e_j) \right] e_i = \frac{k_i}{\left| k \right|}, \quad \left| k \right| = \sqrt{k_1^2 + k_2^2}.
\]

In this equation, \( \mu \) is the shear modulus of the medium, \( \eta = \nu / (1-\nu) \), and \( \nu \) is the Poisson ratio. Note that the integral in eq (3) converges in the ordinary sense.