On the number of Galois points for a plane curve in positive characteristic, III

Satoru Fukasawa

Received: 24 October 2008 / Accepted: 18 September 2009 / Published online: 4 November 2009
© Springer Science+Business Media B.V. 2009

Abstract We consider the following problem: For a smooth plane curve $C$ of degree $d \geq 4$ in characteristic $p > 0$, determine the number $\delta(C)$ of inner Galois points with respect to $C$. This problem seems to be open in the case where $d \equiv 1 \mod p$ and $C$ is not a Fermat curve $F(p^e + 1)$ of degree $p^e + 1$. When $p \neq 2$, we completely determine $\delta(C)$. If $p = 2$ (and $C$ is in the open case), then we prove that $\delta(C) = 0$, $1$ or $d$ and $\delta(C) = d$ only if $d - 1$ is a power of 2, and give an example with $\delta(C) = d$ when $d = 5$. As an application, we characterize a smooth plane curve having both inner and outer Galois points. On the other hand, for Klein quartic curve with suitable coordinates in characteristic two, we prove that the set of outer Galois points coincides with the one of $\mathbb{F}_2$-rational points in $\mathbb{P}^2$.

Keywords Galois point · Plane curve · Positive characteristic

Mathematics Subject Classification (2000) 14H50

1 Introduction

Let the base field $K$ be an algebraically closed field of characteristic $p \geq 0$ and let $C \subset \mathbb{P}^2$ be a smooth plane curve of degree $d \geq 4$. In 1996, Yoshihara introduced the notion of Galois point (see [14, 17] or survey paper [6]). If the function field extension $K(C)/K(\mathbb{P}^1)$, induced from the projection $\pi_P : C \to \mathbb{P}^1$ from a point $P \in \mathbb{P}^2$, is Galois, then the point $P$ is said
to be Galois. When a Galois point $P$ is contained in $C$ (resp. $\mathbb{P}^2 \setminus C$), we call $P$ an inner (resp. outer) Galois point. It is an interesting problem to determine the distribution of Galois points. We denote by $\delta(C)$ (resp. $\delta'(C)$) the number of inner (resp. outer) Galois points for $C$. Yoshihara proved that $\delta(C) = 0$, $1$ or $4$ and $\delta'(C) = 0$, $1$ or $3$ in characteristic $p = 0$ ([14,17]). In characteristic $p > 0$, the notion of Galois points may be sometimes related to that of rational points. Homma [12] proved that the set of Galois points for a Hermitian curve $H : X^2Z + XZ^d - Y^{q+1} = 0$, where $q$ is a power of $p$, coincides with the one of $\mathbb{F}_{q^2}$-rational points in $\mathbb{P}^2$, in particular, $\delta(H) = q^3 + 1$ and $\delta'(H) = q^4 - q^3 + q^2$. Note that the Hermitian curve is projectively equivalent to the Fermat curve $F(q + 1) : X^{q+1} + Y^{q+1} + Z^{q+1} = 0$. Recently, the present author generalized Yoshihara’s result to arbitrary characteristic in most cases ([3–5]). Let $d - 1 = p^e l$ (resp. $d = p^e l$), where $l$ is not divisible by $p$. Combining the results of Homma with the present author, the number $\delta(C)$ (resp. $\delta'(C)$) has been settled except for the cases where $e \geq 2$ and $l = 1$, and where $e \geq 1$, $l \geq 2$ and $l$ divides $p^e - 1$. Such excepted cases appear to be still open ([6, Problem 2]).

The first purpose of this paper is to determine completely the number $\delta(C)$ except for the case where $p = 2$ and $l = 1$, and settle the possible number in the excepted case.

**Theorem 1** Let $C$ be a smooth plane curve of degree $d \geq 4$ in characteristic $p > 0$. Let $d - 1 = p^e l$, where $l$ is not divisible by $p$. Assume that $e \geq 1$ and $C$ is not projectively equivalent to a Fermat curve $F(p^e + 1)$. Then, we have the followings:

1. If $p \geq 3$ or $l \geq 2$, then $\delta(C) = 0$ or $1$.
2. If $p = 2$ and $l = 1$, then $\delta(C) = 0$, $1$ or $d$.

When $p = 2$ and $d = 5$, we will have an example of a smooth curve with $\delta(C) = d$:

**Example** Let $p = 2$ and let $C$ be a curve of degree $d = 5$ defined by

$$g(x, y) := x^4 + ax^2 y + xy^3 + x + y^5 = 0,$$

where $a^2 + a + 1 = 0$. Then, $C$ is smooth and $\delta(C) = 5$.

Summarizing the results of Yoshihara, Homma, the present author and *Theorem 1*, we have the following:

**Theorem 2** Let $C$ be a smooth plane curve of degree $d \geq 4$ in characteristic $p \geq 0$. Then, $\delta(C) = 0$, $1$, $d$ or $(d - 1)^3 + 1$. Furthermore, we have:

1. $\delta(C) = (d - 1)^3 + 1$ if and only if $p > 0$, $d = p^e + 1$ and $C$ is projectively equivalent to the Fermat curve $F(d)$.
2. $\delta(C) = d = 4$ if and only if $p \neq 2, 3$ and $C$ is projectively equivalent to a curve defined by $X^3 Z + Y^4 + Z^4 = 0$.
3. $\delta(C) = d \geq 5$ only if $p = 2$ and $d = 2^e + 1$.

The only remaining open problem on the determination of $\delta(C)$ will become the following:

**Problem** We consider the case where $p = 2$, $e \geq 2$ and $l = 1$. Find and classify smooth plane curves $C$ of degree $d = 2^e + 1$ with $\delta(C) = d$.

The second purpose is to characterize a plane curve having both inner and outer Galois points.

**Theorem 3** Let $C$ be a smooth plane curve of degree $d \geq 4$ in characteristic $p \geq 0$. Let $d - 1 = p^e l$ (resp. $d = p^e l$), where $l \equiv 0 \mod p$ and $l|(p^e - 1)$, if $p > 0$ and $d \equiv 1$ (resp. $d \equiv 0$) mod $p$. Then, $\delta(C) \geq 1$ and $\delta'(C) \geq 1$ if and only if $C$ is projectively equivalent to a curve defined by one of the followings: