The smooth structure set of $S^p \times S^q$

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Abstract We calculate $S^{\text{Diff}}(S^p \times S^q)$, the smooth structure set of $S^p \times S^q$, for $p, q \geq 2$ and $p + q \geq 5$. As a consequence we show that in general $S^{\text{Diff}}(S^{4j-1} \times S^{4k})$ cannot admit a group structure such that the smooth surgery exact sequence is a long exact sequence of groups. We also show that the image of the forgetful map $S^{\text{Diff}}(S^{4j} \times S^{4k}) \to S^{\text{Top}}(S^{4j} \times S^{4k})$ is not in general a subgroup of the topological structure set.

Keywords Smooth structure set · Surgery exact sequence · Product of spheres · Diffeomorphism classification

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1 Introduction

We work in the categories of closed, oriented, simply-connected $\text{Cat}$-manifolds $M$ and $N$ of dimension $n \geq 5$ and orientation preserving maps, where $\text{Cat} = \text{Diff}$ for smooth manifolds or $\text{Cat} = \text{Top}$ for topological manifolds. The $\text{Cat}$-structure set of $M$, $S^{\text{Cat}}(M)$, is the set of structure invariants $[N, f]$ which are equivalence classes of homotopy equivalences $f : N \to M$ where $f_0 : N_0 \to M$ and $f_1 : N_1 \to M$ are equivalent if $f_1^{-1} \circ f_0$ is homotopic to an isomorphism (diffeomorphism or homeomorphism). The base point of $S^{\text{Cat}}(M)$ is the equivalence class of $\text{Id} : M = M$. There is an obvious forgetful map $F : S^{\text{Diff}}(M) \to S^{\text{Top}}(M)$ and a commutative diagram whose rows, respectively the smooth and topological surgery exact sequence for $M$, are long exact sequences of pointed sets:

$$
\begin{align*}
\cdots \quad N^{\text{Diff}}(M \times [0, 1]) &
\xrightarrow{L_{n+1}(e)} N^{\text{Diff}}(M) \xrightarrow{\eta^{\text{Diff}}} N^{\text{Diff}}(M) \xrightarrow{\rho^{\text{Diff}}} L_n(e) \\
\downarrow F &
\downarrow F \quad \downarrow F \\
\cdots \quad N^{\text{Top}}(M \times [0, 1]) &
\xrightarrow{L_{n+1}(e)} N^{\text{Top}}(M) \xrightarrow{\eta^{\text{Top}}} N^{\text{Top}}(M) \xrightarrow{\rho^{\text{Top}}} L_n(e).
\end{align*}
$$

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Here $N^{Cat}(M)$ and $N^{Cat}(M \times [0, 1])$ are the $Cat$ normal invariant sets of $M$ and $M \times [0, 1]$ relative boundary, $L_{n+1}(e)$ and $L_n(e)$ are the surgery obstruction groups: $L_i(e) \cong \mathbb{Z}, 0, \mathbb{Z}_2, 0$ as $i \equiv 0, 1, 2, 3$ mod 4 and the group $L_{n+1}(e)$ acts on $S^{Cat}(M)$ with orbits the fibres of $\eta^{Cat}$. Using identity maps as base points we have Sullivan’s familiar identifications $N^{Cat}(M) \equiv [M, G/Cat]$ and $N^{Cat}(M \times [0, 1]) \equiv [\Sigma M, G/Cat]$ where $\Sigma M$ is the suspension of $M$. We refer the reader to Sect. 3 for some further definitions and references to the literature.

It is a feature of topological surgery that $S^{Top}(M)$ and $N^{Top}(M)$ admit abelian group structures such that the topological surgery exact sequence is a long exact sequence of groups (see Sect. 1.1 for more on this point). In this paper we show that this cannot be the case in general for the smooth category by calculating $S^{Diff}(S^p \times S^q)$ for $p, q \geq 2$ and $n = p + q \geq 5$. We develop the necessary preliminaries by first recalling (1) when $M = S^n$ and (1)$^{Top}$ when $M = S^p \times S^q$.

The Generalised Poincaré Conjecture, due to [28], asserts that $S^{Top}(S^n) = \{[\text{Id}]\}$ whereas the smooth structure set of $S^n, S^{Diff}(S^n) = \Theta_n \equiv \pi_n(Top/O)$, is the finite abelian group of diffeomorphism classes of homotopy $n$-spheres. With $N^{Diff}(S^n) \equiv \pi_n(G/O)$ and $N^{Top}(S^n) \equiv \pi_n(G/Top)$ we have the commuting diagram whose rows are long sequences of abelian groups, the upper row due to [10]:

$$
\begin{align*}
\cdots & \rightarrow \pi_{n+1}(G/O) \xrightarrow{\theta^{Diff}} L_{n+1}(e) \xrightarrow{\eta^{Diff}} \pi_{n}(G/O) \xrightarrow{\theta^{Diff}} L_{n}(e) \\
\cdots & \rightarrow \pi_{n+1}(G/Top) \xrightarrow{\theta^{Top}} L_{n+1}(e) \xrightarrow{\eta^{Top}} \pi_{n}(G/Top) \xrightarrow{\theta^{Top}} L_{n}(e).
\end{align*}
$$

The topological sequence gives the fundamental identification $\pi_n(G/Top) = L_n(e)$ which we often make without further comment. The image of $\omega^{Diff}$ is $bP_{n+1}$, the finite cyclic group of diffeomorphism classes of homotopy spheres bounding parallelisable manifolds: thus $bP_{n+1} \cong L_{n+1}(e)/\theta^{Diff}(\pi_{n+1}(G/O)) \cong \pi_{n+1}(G/Top)/\eta(\pi_{n+1}(G/O))$.

Now let $i : S^p \vee S^q \rightarrow S^p \times S^q$ be the inclusion and $c : S^p \times S^q \rightarrow S^{p+q}$ be the collapse map. We have the identification $N^{Cat}(S^p \times S^q) \equiv [S^p \times S^q, G/Cat]$ where the latter set has a group structure as $G/Cat$ is an infinite loop space. Hence we also have the split exact sequence of abelian groups

$$0 \rightarrow \pi_{p+q}(G/Cat) \xrightarrow{i^*} [S^p \times S^q, G/Cat] \xrightarrow{i^*} \pi_p(G/Cat) \times \pi_q(G/Cat) \rightarrow 0.\$$

In Sect. 4 (13), based on Lemma 3.2, we use the product of normal maps to define a section $\pi_{p,q}$ to $i^*$ and thus an identification

$$\pi_{p,q} \oplus c^* : \prod_{i=1}^3 \pi_{p_i}(G/Cat) \cong [S^p \times S^q, G/Cat], \quad (3)$$

where $p_1 = p$, $p_2 = q$ and $p_3 = p + q$.

In the topological case, see [20][Ex. 20.4] and [12][§7], the map $i^* \circ \eta^{Top}$ defines a bijection

$$i^* \circ \eta^{Top} : S^{Top}(S^p \times S^q) \equiv \pi_p(G/Top) \times \pi_q(G/Top). \quad (4)$$

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