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Universal hyperbolic geometry I: trigonometry

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Abstract Hyperbolic geometry is developed in a purely algebraic fashion from first principles, without a prior development of differential geometry. The natural connection with the geometry of Lorentz, Einstein and Minkowski comes from a projective point of view, with trigonometric laws that extend to 'points at infinity', here called 'null points', and beyond to 'ideal points' associated to a hyperboloid of one sheet. The theory works over a general field not of characteristic two, and the main laws can be viewed as deformations of those from planar rational trigonometry. There are many new features; this paper gives 92 foundational theorems.

Keywords Hyperbolic geometry · Projective geometry · Rational trigonometry · Relativistic geometry · Null points

Mathematics Subject Classification (2000) 14N99 · 53A35 · 51F99

1 Introduction

Hyperbolic geometry is set out here in a new and completely algebraic way. This view of the subject, called *universal hyperbolic geometry,* is a special case of the more general geometry described in [26], and has the following characteristics that generally distinguish it from classical hyperbolic geometry, found in for example [1, 3, 5, 9, 12, 15], or from other approaches to the subject, such as [14, 17] or [22].

- A more direct and intimate connection with the geometry of Einstein's Special Theory of Relativity in the framework of Lorentz and Minkowski. In fact *hyperbolic geometry is precisely projective relativistic geometry.* This is a fundamental understanding. The connection with relativistic geometry is also a key feature of [22].
- The basic set-up allows a consistent development of hyperbolic geometry *over the rational numbers*, and *over a finite field*. This ties in with work of [2, 18, 20] and others.

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- A crucial *duality* between points and lines that connects with, and clarifies, the *pole-polar* duality with respect to the unit circle of projective geometry.
- An unambiguous and concrete treatment of what are traditionally called ‘points at infinity’, here called *null points,* together with what former generations of projective geometers called ‘ideal points’ which lie on *null lines*. In terms of relativistic geometry, we study the hyperboloid of two sheets with equation $x^2 + y^2 - z^2 = -1$, the null cone with equation $x^2 + y^2 - z^2 = 0$, and the hyperboloid of one sheet with equation $x^2 + y^2 - z^2 = 1$ *together*. The relevance of the latter is also discussed in [21].
- The fundamental metrical measurements of *quadrance* between points and *spread* between lines are similar to the corresponding notions in planar rational trigonometry, and the basic laws of hyperbolic trigonometry may be seen as deformations of those of planar rational trigonometry (see [23]).
- Transcendental functions, such as $\log x$, $\sinh x$ or $\cos x$ are not needed. A prior development of the real number system is not needed, so the logical difficulties often associated with this topic are avoided.
- The existence of a rich *null trigonometry*: trigonometric relations that involve null points and null lines.
- *Parallels* play a more specialized role. Somewhat ironically, *Euclid’s parallel postulate holds*.
- The isometry group of the geometry *does not act transitively* on the space. The universal hyperbolic plane has aspects which appear in negatively curved Riemannian geometry, other aspects which are Lorentzian, and other aspects which are Euclidean.
- The framework is *algebraic geometry*, rather than *differential geometry*. But it is a form of algebraic geometry that relates more to the historical approach prior to the twentieth century direction. The focus is on *metrical relations* and *concrete polynomial identities* which *encode geometric realities*.

1.1 Advantages of the new approach

The advantages of universal hyperbolic geometry over the classical approach include:

- *Simplicity and elegance:* the subject is simple enough to be accessible to beginning undergraduates and motivated high school students without a prior understanding of calculus or real numbers. The elementary aspects fit together pleasantly.
- *Logical clarity:* traditional treatments of hyperbolic geometry often have obscure foundations, or are rife with arguments that rely on pictorial understanding. The purely algebraic framework frees us from logical difficulties, and allows us to aspire to a complete and unambiguous treatment of the subject from first principles.
- *Greater accuracy in computations:* many problems can now be solved completely, whereas the classical theory provides only approximate solutions.
- Connections with *number theory* become more explicit.
- *New directions for special functions:* The remarkable *spread polynomials* of planar rational trigonometry also play a key role in hyperbolic geometry. This family of (almost) orthogonal polynomials replaces the *Chebyshev polynomials of the first kind*.
- *Extension of classical geometry:* many more traditional results of Euclidean geometry can now be given their appropriate hyperbolic analogs. Of the thousands of known results in Euclidean geometry, only a fraction currently have analogs in the hyperbolic setting. This turns out to be a consequence of the way we have, up to now, viewed the subject. *Hyperbolic geometry is a richer and ultimately more important theory than Euclidean geometry*.