On some foliations arising in $\mathcal{D}$-module theory

S. C. Coutinho

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Abstract We describe the properties of some foliations which arise in the study of the characteristic variety of $\mathcal{D}$-modules constructed from vector fields of an affine space.

Keywords Derivation · Singularity · Invariant variety · Hamiltonian · Symplectic geometry · $\mathcal{D}$-module

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1 Introduction and motivation

In a paper [14] of 1878 Darboux proposed a method for finding a first integral of a differential equation in terms of the algebraic curves tangent to the vector field that defines that equation, the invariant algebraic curves of the vector field. Darboux also pointed out the importance of studying the singularities of the differential equation to the analysis of the invariant algebraic curves. Darboux’s ideas were taken up in the nineteenth century by Poincaré and have recently flourished in the work of several mathematicians, among them Jouanolou [23], Cerveau and Lins Neto [6], Carnicer [5] and Walcher [31].

Using the language of algebraic geometry we may generalize invariant algebraic curves to higher dimensional varieties. Let $X$ be a smooth complex algebraic variety over which a one dimensional foliation $\mathcal{F}$ has been defined. Such a foliation corresponds to a map $f : \Omega^1_X \to \mathcal{L}$,
where $\Omega^1_X$ is the sheaf of Kähler differentials and $\mathcal{L}$ is a line bundle over $X$. Dualizing this sequence and tensoring it up with $\mathcal{L}^{-1}$, we get a homomorphism $\mathcal{O}_X \rightarrow \mathcal{L}^{-1} \otimes \Theta_X$. Thus, the foliation $\mathcal{F}$ may also be defined by a section of the sheaf $\mathcal{L}^{-1} \otimes \Theta_X$. A point $x \in X$ is a singularity of $\mathcal{F}$ if $f$ is not surjective at $x$. The set of all singularities of $\mathcal{F}$ will be denoted by $\text{Sing}(\mathcal{F})$. A subscheme $Y$ of $X$ is invariant under $f$ if there exists a map $\Omega^1_Y \rightarrow \mathcal{L}|_Y$ such that the diagram

$$
\begin{array}{c}
\Omega^1_X|_Y \\
\downarrow \\
\Omega^1_Y
\end{array} \xrightarrow{f|_Y} \begin{array}{c}
\mathcal{L}|_Y \\
\end{array}
$$

is commutative. For more details see [13]. The study of invariant algebraic subvarieties in this more general setting has been considered by Soares [28,29], Esteves [17], and Esteves and Kleiman [18,19], among others.

In this note we study the singularities (Sect. 2) and invariant algebraic subvarieties (Sect. 3) of foliations of $\mathbb{P}^n \times \mathbb{P}^n$ induced by hamiltonian vector fields determined by bihomogeneous polynomial functions of $\mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$ that are linear on the $y$s. The motivation for studying this special case comes from the interplay between symplectic geometry and the theory of $\mathcal{D}$-modules.

A $\mathcal{D}$-module is a module over a ring of differential operators of a smooth complex algebraic variety $X$. Since these rings are noncommutative, their modules often have a very rich structure, which can be studied with the help of an important geometric invariant, the characteristic variety, which is a subvariety of the cotangent bundle $T^*X$. The cotangent bundle has a natural symplectic structure, relative to which the characteristic variety of a $\mathcal{D}(X)$-module has to be co-isotropic; see [20] or [8] for more details.

The most important special case of this construction is arguably that of the ring of differential operators of the complex affine space $\mathbb{A}^n$. As has been shown in [3,9,10,15] and [30], quotients of these rings by cyclic left ideals generated by operators of order one are an excellent source of examples of $\mathcal{D}(\mathbb{A}^n)$-modules with various interesting properties. It turns out that such modules have for their characteristic varieties hypersurfaces defined by polynomials that are linear in the fibres.

More precisely, if $x_1, \ldots, x_n$ are coordinates of $\mathbb{A}^n$ and $y_1, \ldots, y_n$ are the corresponding conjugate coordinates on the fibres of $T^*\mathbb{A}^n$, then these polynomials can be written in the form $f = \sum_{i=1}^n a_i y_i$, where $a_i \in \mathbb{C}[x_1, \ldots, x_n]$ for $1 \leq i \leq n$. In this case, the hamiltonian vector field $\xi_f$ induced by $f$ has the form given in Eq. (2.1). The co-isotropy implies that the characteristic variety of any submodule or quotient of a module whose characteristic variety has equation $f = 0$ is invariant under $\xi_f$.

By construction, these characteristic varieties are always conical, that is homogeneous with respect to the $y$s. So, introducing a new variable $x_0$, we can homogenize both $\xi_f$ and $f$ with respect to the $x$s. The resulting vector field of $\mathbb{C}^{n+1} \times \mathbb{C}^n$ induces a foliation in $X = \mathbb{P}^n \times \mathbb{P}^{n-1}$, which leaves the hyperplane $x_0 = 0$ invariant. Since this hyperplane is naturally isomorphic to $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$, the foliation that £ induces on it is an example of the kind of foliation we propose to study in this note. This particular foliation plays an important rôle in the solution of a case of a conjecture of Bernstein and Lunts, see [11].

For another interesting example, we turn to conormal varieties. Keeping the notation above for the coordinates, let $I$ be a homogeneous ideal of $\mathbb{C}[x_1, \ldots, x_n]$ and consider the conormal variety $Y$ with support on $Z \subset \mathbb{A}^n$, the variety of zeroes of $I$. In other words,