Abstract  The quantum theory of angular momentum affords a treatment of tensors and vectors in a spherical basis. By using this theory we define the tensor differential operators: divergence, curl and gradient which act on a tensor of any rank, in terms of C-G coefficients. With these definitions we obtain a matrix representation and useful properties for those operators. An interesting application of this formalism is to find the wave equation of a tensor of any rank in a linear theory. This provides a new common way to look at the wave equations associated with both Maxwell’s equations and the Maxwell-like equations for the linearized Weyl curvature tensor in gravitoelectromagnetism describing gravitational radiation on a Minkowski spacetime background.

Keywords  Differential operators · C-G coefficients · Gravito-electric and gravito-magnetic fields · Wave equation · Gravitoelectromagnetism

1 Introduction

The standard definitions of the vector differential operators divergence, curl and gradient are well-known in cartesian coordinates. Those operators appear in many equations of physics. It is very useful to express these differential operators in terms of C-G coefficients.

The ladder operators of quantum mechanics lead us to introduce the spherical basis [1] which is more convenient for describing vectors and tensors since they can be easily expressed in their irreducible forms in this basis. An irreducible tensor [1, 2, 5], which is traceless and symmetric, of rank \( l \) has \( 2l + 1 \) components and they have the same transformation properties under rotation as the spherical harmonics of rank \( l \).
We may construct expressions for the components of any quantity which transforms like a vector under rotations. With the aid of the spherical basis we may use the coupling methods \([1, 3]\) of angular momenta to construct spherical tensors (irreducible tensors) of any rank from the spherical components of a given set of vector quantities. Thus, we use the coupling method for two angular momenta and the definition of tensor product to obtain these differential operators in terms of the C-G coefficients. This provides us with a matrix representation and some properties for these operators. In a linear theory, the field equations of gravitation in a vacuum with sources have the same form as Maxwell’s equations of electromagnetism. Tensors of rank one describe electromagnetism and tensors of rank two describe gravitation. By using these field equations and the properties of the differential operators we can obtain the wave equation for the electromagnetic and linearized gravitational interactions on a Minkowski spacetime background. This provides the group theoretical explanation for the close analogy between electromagnetism and linearized gravitoelectromagnetism. We generalize this treatment for a tensor of any rank and obtain its wave equation.

In this work we only deal with irreducible tensors \(T = T_{(abc...)}\) (cartesian) or \(T_{m}^{(l)}\) (spherical ) because they are the simplest representations and all others can be built up from them.

The operator nabla \(\nabla = (\partial_x, \partial_y, \partial_z)\) will be involved in all our definitions. Its components in the spherical basis are \(D^1 = (\partial_1, \partial_0, \partial_{-1})\). Both sets of components are related as follows

\[
\partial_1 = \partial_1^{(1)} = -\frac{1}{\sqrt{2}}(\partial_x + i\partial_y), \quad \partial_0 = \partial_0^{(1)} = \partial_z, \quad \partial_{-1} = \partial_{-1}^{(1)} = \frac{1}{\sqrt{2}}(\partial_x - i\partial_y).
\]

2 Differential operators

In \(R^3\) the divergence constructs a scalar field from a vector field, the curl a vector field from a vector field and the gradient a vector field from a scalar field. The definition of these operators can be extended to tensors of rank \(l\) that are symmetric and traceless (irreducible). The set of all symmetric-trace-free Cartesian tensor of rank \(l\) generates an irreducible representations of rank \(l\) and dimension \(2l + 1\) of the group of proper rotations \(SO(3)\). Thus the tensors \(T_{(abc...)}\) and \(T_{m}^{(l)}\), both of rank \(l\) have the same number of components \(2l + 1\). The individual components \(T_{m}^{(l)}\) can be expressed as linear combination of the rank \(l\) cartesian tensor components \(T_{(abc...)}\) and vice versa.

2.1 Divergence

The divergence operator (div) constructs a tensor field of one lower rank from a tensor field. We can define this operator on a tensor of rank \(l\) in two ways.