Analytical solutions for black-hole critical behaviour

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Abstract  Dynamical Einstein cluster is a spherical self-gravitating system of counter-rotating particles, which may expand, oscillate and collapse. This system exhibits critical behaviour in its collapse at the threshold of black hole formation. It appears when the specific angular momentum of particles is tuned finely to the critical value. We find the unique exact self-similar solution at the threshold. This solution begins with a regular surface, involves timelike naked singularity formation and asymptotically approaches a static self-similar cluster.

General relativistic numerical simulation (numerical relativity) has revealed critical phenomena at the threshold of black hole formation in self-gravitating systems [1]. When a parameter $p$, which parametrises a generic one-parameter family of initial data sets, is tuned to the critical value $p^*$, there appears a self-similar solution, which is called a critical solution. Beyond this value, the collapse ends in a black hole, its mass $M_{\text{BH}}$ obeying the power law $M_{\text{BH}} \propto |p - p^*|^{\gamma}$, where $\gamma$ is called a critical exponent. The critical behaviour is well described in terms of the behaviour of solutions around a self-similar solution with a single unstable mode [2]. In this approach, self-similar solutions with regularity conditions are numerically found and a self-similar solution is numerically shown to be with a single, linearly unstable mode. See [3] for a recent review of critical phenomena. Apparently, there still is a huge gap between numerical
simulation and linear stability analysis. Moreover, one could suspect unresolved fine structure at the threshold because almost all results have been based on numerics with finite accuracy (cf. [4]).

Here we show that there is a system where we can discuss critical phenomena in an analytical and exact manner. This is the spherical system of counterrotating particles, first introduced by Einstein [5] and later generalised to a dynamical case [6–8]. Using a coordinate \( r \) comoving to the radial motion of each shell, the line element in this spacetime is given by

\[
d s^2 = -e^{2\nu(t,r)} dt^2 + e^{2\psi(t,r)} dr^2 + R^2(t, r)d\Omega^2,
\]

where \( d\Omega^2 \) is the line element on the 2D unit sphere. The Einstein equations and conservation law reduce

\[
\nu' = -\frac{1}{h(r, R)} \frac{\partial h(r, R)}{\partial R} R', \quad e^{2\psi} = \frac{(R')^2 h^2(r, R)}{1 + 2E(r)},
\]

and

\[
\dot{R}^2 e^{-2\nu} = -1 + \frac{2M(r)}{R} + \frac{1 + 2E(r)}{h^2(r, R)},
\]

where \( \dot{\equiv} \partial/\partial t \) and \( ' \equiv \partial/\partial r \), \( M(r) \) and \( 2E(r) > -1 \) are arbitrary functions corresponding to the Misner–Sharp mass and the specific energy, respectively, and \( h(r, R) \) is given by

\[
h^2(r, R) = 1 + \frac{L^2(r)}{R^2},
\]

where \( L(r) \) is the specific angular momentum of counterrotating particles. The energy density is

\[
\epsilon = \frac{M'}{4\pi R^2 R'}.
\]

The motion of each shell is governed by Eq. (3) or

\[
\frac{1}{2} \left( \frac{dR}{d\tau} \right)^2 + U(r, R) = E(r),
\]

where \( d\tau = e^\nu dt \) and

\[
U(r, R) \equiv -\frac{M(r)}{R} + \frac{(1 + 2E(r))L^2(r)}{2(R^2 + L^2(r))}.
\]

If we assume that the solution has a regular surface on which all regular metric functions and physical quantities are also analytic, this implies the Taylor-series expandability in terms of \( R^2 \). The arbitrary functions \( M, E \) and \( L^2 \) then should be expanded as \( M = M_3 r^3 + M_5 r^5 + \cdots \), \( E = E_2 r^2 + E_4 r^4 + \cdots \), and \( L^2 = L_4 r^4 + L_6 r^6 + \cdots \), if we choose \( r \) so that \( r = R \) on the initial regular surface. \( R \) is