Falling into a Schwarzschild black hole
Geometric aspects

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Abstract Consider a radially freely falling observer who plunges into a Schwarzschild black hole. In contrast to a static observer, he will have a different view of the black hole and of the outer sky. Furthermore, the relationship between the proper time of the falling observer and the proper time of a distant static observer differs from the relationship between the proper times of two static observers or two freely falling observers.

Keywords Schwarzschild black hole · Local observer · Proper time · Apparent size of a black hole

1 Introduction

In this article we will follow two questions from a pedagogical way: how big does a non-rotating black hole appear to a static observer in contrast to a freely falling one and what is the relationship between the proper times of two arbitrary observers. In both situations special and general relativistic effects compete with each other. The apparent size of the black hole depends on the bending of light close to the black hole and the aberration of light of the fast moving observer, whereas the relationship between the proper times follows from the time dilation due to the curved spacetime and due to the motion of the observer. As long as we expect that each black hole forms from a collapsing star, a distant observer will never see a true black hole because at least the surface of the star appears to never reaching the event horizon. A freely falling observer outside the black hole also argues that the surface of the collapsing star only approximately reaches the horizon. However, if he drops below the horizon, he will
completely change his minds. On the other hand, an observer at rest with respect to the freely falling surface of the collapsing star will not see something extraordinary until he crashes into the singularity.

Carter [4] discusses the apparent size of a black hole for an observer who falls freely from “infinity”. Bakala et al. [3] concentrate on the apparent size of the more general Schwarzschild-de Sitter black hole depending on different values for the cosmological parameter $\Lambda$. We will generalize the situation of an observer falling freely into a Schwarzschild black hole to an arbitrary initial point outside the black hole. The reciprocal problem of the appearance of distant objects to an observer near a black hole or a freely falling observer is discussed by Cunningham [6]. Schee et al. [21] concentrate on the object’s point of view and discuss by means of the effective potential whether null geodesics in Kerr spacetime are able to reach the distant observer or fall below the horizon. Oppenheimer and Snyder [19] investigated the gravitational contraction of a star represented by a ball of dust in comoving coordinates. The optical appearance of such a star that is collapsing through its gravitational radius is given by Ames and Thorne [2]. Jaffe [11] analysis the backward emission of light from a collapsing object. Grave [10] and Müller [17] studied the visual appearance of the Oppenheimer–Snyder collapse by four-dimensional raytracing.

The Schwarzschild spacetime can be represented by numerous different coordinates like Schwarzschild, Eddington–Finkelstein [7, 9], Kruskal [14], Novikov [18] or Painlevé–Gullstrand [15] coordinates. Each of them can be used to stress different features of the spacetime or to simplify some calculations. In Sect. 2, we will review the Schwarzschild spacetime in Eddington–Finkelstein coordinates which are best suited for describing ingoing light rays. We also introduce the natural local tetrad of a static observer which serves as his local reference frame. In contrast to Schwarzschild coordinates, the local tetrad of a freely falling observer in Eddington–Finkelstein coordinates is regular below the horizon as we will see in Sect. 3. In Sect. 4, we discuss the apparent size of the black hole depending on the current position and velocity of the falling observer. In Sect. 5 we compare how a distant, static observer and a freely falling one perceive the progress of time of each other. In Sect. 6 we study what two freely falling observers will see from each other.

2 The Schwarzschild metric in Eddington–Finkelstein coordinates

In general, the Schwarzschild spacetime is being described by means of the well known Schwarzschild coordinates $x^\mu = (t, r, \vartheta, \varphi)$. Because the Schwarzschild metric has a coordinate singularity at the horizon the corresponding metric is not appropriate for the investigation of an observer falling into the black hole. A better choice are the coordinates found by Eddington [7] and Finkelstein [9]. They have introduced a new coordinate such that ingoing or outgoing radial null geodesics are described in a simple way. The ingoing Eddington–Finkelstein metric with the advanced null coordinate $v$ is given by the line element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$,

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 ds^2 = - \left( 1 - \frac{r_s}{r} \right) c^2 dv^2 + 2c \, dv \, dr + r^2 d\Omega^2,
$$

(1)