Periodic Quasi-Exactly Solvable Models

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Various quasi-exact solvability conditions, involving the parameters of the periodic associated Lamé potential, are shown to emerge naturally in the quantum Hamilton–Jacobi (QHJ) approach. We study the singularity structure of the quantum momentum function, which yields the band-edge eigenvalues and eigenfunctions and compare it with the solvable and quasi-exactly solvable non-periodic potentials, as well as the periodic ones.

KEY WORDS: quasi-exactly solvable Hamiltonians; quantum Hamilton–Jacobi formalism; associated Lamé potential.

1. INTRODUCTION

Quasi-exactly solvable (QES) Hamiltonians, interconnecting a diverse array of physical problems, have been the subject of extensive study in recent times (Atre and Panigrahi, 2003; Geogo et al., 2003; González-López et al., 1991; Razavy, 1980, 1981; Shifman, 1989; Singh et al., 1978; Turbiner and Ushveridze, 1987; Ushveridze, 1994; Znojil, 1983). These systems, containing a finite number of exactly obtainable eigenstates, have been linked with classical electrostatic problems, as also to the finite dimensional irreducible representations of certain algebras. Some of these studies employ group theoretical methods, others are based on the symmetry of the relevant differential equations (González-López et al., 1991; Shifman, 1989; Turbiner and Ushveridze, 1987; Ushveridze, 1994). The key to the existence of the finite number of identifiable states is the quasi-exact solvability condition, relating certain potential parameters of these dynamical systems. An interesting feature, distinguishing these QES systems from known exactly solvable cases, is the presence of complex zeros in the polynomial part of the wave functions. Hence, quantum Hamilton–Jacobi (QHJ) formalism, being naturally formulated in the complex domain, is ideally suited for studying the
QES problems (Geogo et al., 2003). Although polynomial potentials have been studied rather exhaustively, QES periodic potentials have not received significant attention in the literature.

The associated Lamé potential (ALP),

\[ V(x) = a(a + 1)m \text{sn}^2(x, m) + b(b + 1)m \frac{\text{cn}^2(x, m)}{\text{dn}^2(x, m)} \]  

is an interesting example of a periodic potential, which is exactly solvable, when \( a = b \) and shows QES property, when \( a \neq b \). Here, \( \text{sn}(x, m) \), \( \text{cn}(x, m) \), and \( \text{dn}(x, m) \) are the doubly periodic elliptic functions with modulus parameter \( m \) (Hancock, 1958). The ALP has a periodic lattice of period \( K(m) \) with the basis composed of two different atoms which are alternately placed. It possesses a surprisingly large variety of QES solvability conditions depending on the nature of the potential parameters \( a \) and \( b \).

For \( a, b \) being unequal integers, with \( a > b > 0 \), there are \( a \) bound bands followed by a continuum band, whose band-edge solutions can be obtained analytically. If \( a - b \) is odd (even) integer, it has \( b \) doubly degenerate band edges of period \( 2K(4K) \), which cannot be obtained analytically. The existence of analytically inaccessible states can be ascertained through the oscillation theorem. For \( a, b \) having half-integral values (with \( a > b \)), there are infinite number of bands with band-edge wave functions having period \( 2K(4K) \), if \( a - b \) is odd (even). Of these infinite number of bands, \( a - b \) bands have band edges, which are non-degenerate with period \( 2K(4K) \) and \( b + \frac{1}{2} \) doubly degenerate states of period \( 2K(4K) \) that can be obtained analytically. For \( a \) being an integer and \( b \) being a half-integer or vice versa, one can obtain some exact analytical results for mid-band states (Khare and Sukhatme, 1999, 2001; Magnus and Wrinkler, 1966).

It is quite natural to enquire about the origin of this rich QES structure in the associated Lamé potential (Ganguly, 2002; Tkachuk and Voznyak, 2002). This paper is devoted to the study of the same through the quantum Hamilton–Jacobi approach. The quasi-exact solvability conditions, involving the parameters of the periodic associated Lamé potential, are shown to emerge naturally. We also study the singularity structure of the quantum momentum function, which yields the band-edge eigenvalues and eigenfunctions. As will be seen in the text, the present approach is quite economical as compared to the earlier known methods, for determining the eigenvalues and eigenfunctions (Arscot, 1964; Magnus and Wrinkler, 1966).

In our earlier studies, we had looked at non-periodic ES, QES, and ES periodic potentials through the QHJ formalism which was initiated by Leacock and Padgett (1983). We were successful in obtaining the quasi-exact solvability condition (Geogo et al., 2003) for QES models and in obtaining the eigenvalues and eigenfunctions for the ES and the band-edge eigenfunctions and eigenvalues of the ES periodic Lamé and the associated Lamé potentials (Sree Ranjani et al.,