Solutions of a Class of Duffing Oscillators with Variable Coefficients

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Received: 20 September 2010 / Accepted: 29 October 2010 / Published online: 18 November 2010 © Springer Science+Business Media, LLC 2010

Abstract The solutions of a class of nonlinear second-order differential equations with a cubic term in the dependent variable being related to Duffing oscillators are obtained by means of the factorization technique. The Lagrangian, the Hamiltonian and the constant of motion are also found through a correspondence with an autonomous system. A physical example is worked out in this frame.

Keywords Duffing equations · Factorization technique

1 Introduction

In the present work we will consider an interesting class of nonlinear second-order ordinary differential equations (ODE) with variable coefficients, which contains a cubic term and will be called in the sequel cubic nonlinear (CNL) equations, of the form

\[ Y''(t) + F_1(t)Y'(t) + 2F_2^2(t)Y^3(t) + F_3(t)Y(t) + F_4(t) = 0, \]

(1)

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where $Y(t)$ is the dependent variable, $F_i(t)$ ($i = 1, \ldots, 4$) are functions of the independent variable $t$, and the subindex $t$ denotes derivative with respect to this argument, $Y_t \equiv dY/dt$. These equations can be seen as a variation of the well known Duffing oscillator

$$\theta_{tt}(t) + a\theta_t(t) + b\theta(t) + c\theta^3(t) + d\cos(\omega t) = 0,$$

(2)

where the coefficient $a$ is related with the damping, $b$ and $c$ are determined by the restoring force, and the last term is the periodic external force. In the literature there are many studies about several equations of the Duffing type [1–8], in which different properties have been examined numerically or analytically.

In general, there are no standard techniques to solve this kind of equations (1) involving variable coefficients. However, when the coefficients are constant they have been investigated using many procedures: the elliptic averaging [9], the sinh-Gordon equation expansion [10], the homotopy-perturbation [11], and others mentioned in [9–11]. In this paper we will deal with this equation by means of the factorization method, which in general is applicable to nonlinear ordinary differential equations (ODE’s) when the coefficients do not depend on the independent variable [12–15]. However, as we shall see, the case of variable coefficients can be also factorized by transforming the initial equation into a canonical form [16].

The structure of this work is the following. We start in Sect. 2 with the scale transformation of the CNL second-order differential equation (1) into its canonical form. In this way, we connect a non-autonomous system to an easier autonomous one. Then, in Sect. 3, we will find the Lagrangian and Hamiltonian for this canonical form as well as for the original differential equation, together with a relevant constant of motion. Next, in Sect. 4 the CNL second-order differential equation and its canonical form are factorized, and the solutions obtained in this way will be given in Sect. 5. We will apply our development along Sect. 6 to an interesting example belonging to this class of non-autonomous equations: a pendulum with time dependent damping and driving force. Section 7 will end this work with some remarks and conclusions.

### 2 Reduction to Canonical Form

As it is well known, the second-order integrable ODE’s were classified by Painlevé [17]. Following the standard procedure, let us make the following scale transformation in the dependent and independent functions

$$dt = \frac{1}{\lambda(t)} \, dz, \quad Y(t) = \frac{1}{\alpha(t)} \, W(z)$$

(3)

in order to reduce (1) into one of the canonical forms of Painlevé’s classification. The auxiliary functions $\lambda(t)$ and $\alpha(t)$ will be found later on in terms of the initial data $F_i(t)$. Indeed, the new dependent variable $W(z)$ is required to satisfy

$$W_{zz} + 2W^3 - Q(z)W + 1 = 0,$$

(4)

where the function $Q(z)$ is such that $Q_{zz} = 0$. When $Q_z \neq 0$, (4) can be easily transformed into the second Painlevé transcendent [18]. Henceforth, we will concentrate on the case $Q_z = 0$, so that (4) takes the form

$$W_{zz} + 2W^3 - c_0 W + 1 = 0,$$

(5)