QUASI-STEADY-STATE FLOWS IN A LIQUID FILM IN A GAS:
COMPARISON OF TWO METHODS OF DESCRIBING WAVES

O. V. Voinov

UDC 532.516

The motion of thin films of a viscous incompressible liquid in a gas under the action of capillary forces is studied. The surface tension depends on the surfactant concentration, and the liquid is nonvolatile. The motion is described by the well-known model of quasi-steady-state viscous film flow. The linear-wave solutions are compared with the solution using the Navier–Stokes equations. Situations are studied where a solution close to the inviscid two-dimensional solutions exists and in the case of long wavelength, the occurrence of sound waves in the film due to the Gibbs surface elasticity is possible. The behavior of the exact solutions near the region of applicability of asymptotic equations is studied, and nonmonotonic dependences of the wave characteristics on wavenumber are obtained.

Key words: viscous incompressible liquid, capillary forces, surface tension, film in a gas, wave dynamics.

1. Model of Quasi-Steady-State Flows in a Film. The thermodynamic foundations of the dynamics of liquid films with variable surface tension were developed in Gibbs papers on interfacial thermodynamics [1]. Gibbs performed important studies of the dynamics of liquid films with a surfactant, which were continued by Mysels et al. [2]. The motion of liquid films in a gas were theoretically and experimentally studied by Taylor [3]. Taylor’s theory of the dynamics of free liquid films and the theory of shallow water on a horizontal surface [4] are based on the ideal fluid model. Unlike in the problems considered in these theories, in the case where the film surface is subjected to tangential stress, liquid viscosity plays an important role [5, 6]. Due to viscosity, the velocity of the liquid can change insignificantly over the film thickness. In a free film, sound waves due to the Gibbs surface elasticity [5] are possible.

We consider a film symmetric about the plane $x_3 = 0$ using the following assumptions. The distance $\lambda$ at which the flow parameters vary is large compared to the film thickness $h$ ($\lambda \gg h$). The characteristic time $\tau$ far exceeds the time of the transfer vortex across the film: $h^2 \ll \nu \tau$ ($\nu$ is the kinematic viscosity). This condition is assumed to be satisfied for the time $\tau$ calculated from the value of $\lambda$ and the characteristic liquid velocity: $\tau = \lambda / \nu^*$, which is equivalent to the smallness of the reduced Reynolds number of the hydrodynamic theory of lubrication: $v_h^2 / (\lambda \nu) \ll 1$. Under these assumptions [6], the flow at each cross-sections of the film is close to steady-state flow with quadratic velocity profile and coefficients that vary slowly over the coordinates $x_1, x_2$:

$$v_i' = v_i(x_1, x_2, t) + \frac{3}{2} \left(1 - \left(\frac{2x_3}{h}\right)^2\right) u_i(x_1, x_2, t), \quad i = 1, 2. \hspace{1cm} (1.1)$$

Here $v_i$ is the velocity of the film surface and $u_i$ is the average flow velocity relative to the surface. In the equations of motion of the film, the acceleration of the liquid is determined primarily by the velocity $v$ [6]. To take into account the effect of the velocity $u$ on the small attenuation of waves in the film [7], we find the acceleration of the liquid from the average value of the velocity $v'$ in the film. The difference between the average acceleration of
the liquid in the film, determined by relation (1.1), and the acceleration calculated from the average velocity $\mathbf{v}$, is proportional to the small quantity $(\mathbf{u} \cdot \nabla) \mathbf{u}$, whose value is insignificant in the adopted film flow model.

The equations of quasi-steady-state film flows [6] are written as

$$\frac{d}{dt} (h \rho v_i) = 2\nabla_i \sigma + h \rho g_i + h \nabla_i \left( \frac{\sigma}{2} \nabla^2 h - p_e \right) + \nabla_j \{ h \mu (2 \delta_{ij} \text{div } \mathbf{v} + \nabla_i v_j + \nabla_j v_i) \}; \quad (1.2)$$

$$6 \mu \mathbf{u} = -h \nabla \sigma; \quad (1.3)$$

$$\frac{\partial h}{\partial t} + \nabla \cdot h \mathbf{v} = 0, \quad \mathbf{v}' = \mathbf{u} + \mathbf{v}; \quad (1.4)$$

$$\sigma = \sigma(\Gamma, T), \quad \Gamma = \Gamma(c, T), \quad x_3 = \pm h/2; \quad (1.5)$$

$$\frac{\partial \Gamma}{\partial t} + \text{div} (\Gamma \mathbf{v}) = -D \frac{\partial c}{\partial x_3} (x_1, x_2, \frac{h}{2}, t); \quad (1.6)$$

$$\frac{\partial c}{\partial t} + v_i' \nabla_j c - (x_3 \text{div } \mathbf{v}) \nabla_3 c = D \nabla^2 c, \quad x_3 \in \left( -\frac{h}{2}, \frac{h}{2} \right). \quad (1.7)$$

Here $\delta_{ij}$ is the Kronecker symbol, the summation is performed over the repeated index $j = 1, 2$, $g_i$ is the body force, $\rho$ is the density, $\mu$ is the dynamic viscosity, $D$ is the diffusion coefficient, $c$ and $\Gamma$ are the volumetric and surface impurity concentrations, whose profiles are symmetric about the point $x_3 = 0$; the concentration $\Gamma$ is related to the value of $c$ on the surface by the adsorption equation (1.5); the surface tension $\sigma$ depends on $\Gamma$ according to the first equation in (1.5); the gas temperature $T$ and pressure are constant; $p_e$ is an additional term in the pressure expression for ultrathin films (differs in sign from the wedging pressure). The expression for the effective pressure $p_e$ takes into account, in particular, the action of Van der Waals forces, which is described by similar equations for a film in a gas and a film on a solid wall [5, 8–13]. Equations (1.6) and (1.7) describe impurity transfer in the liquid. The possible effect of the surface rheology [6] is not considered.

There is divergent form of the momentum equation (1.2), which is suitable for writing the integrals of the equations [14]:

$$\frac{\partial}{\partial t} (h \rho v_i') + \nabla_j I_{ij} = h \rho g_i, \quad I_{ij} = \rho h v_i' v_j' - 2s_{ij} - h p_{ij} - E \delta_{ij},$$

$$s_{ij} = \sigma(1 + |\nabla z|^2/2) \delta_{ij} - \sigma(\nabla_i z) \nabla_j z, \quad z = h/2,$$

$$p_{ij} = \delta_{ij} ((\sigma/2) \nabla^2 h - p_e) + \mu (2 \delta_{ij} \text{div } \mathbf{v} + \nabla_i v_j + \nabla_j v_i).$$

Here $v_i'$ is the average velocity and $E$ is the primitive of $p_e = \partial E/\partial h$.

In the case of an insoluble impurity in a liquid, Eqs. (1.5)–(1.7) reduce to the two equations

$$\sigma = \sigma(\Gamma, T), \quad \frac{\partial \Gamma}{\partial t} + \text{div} (\Gamma \mathbf{v}) = 0. \quad (1.8)$$

Using (1.8), we write the surface elasticity coefficient as

$$\varepsilon' = -\Gamma \frac{\partial \sigma}{\partial \Gamma}. \quad (1.9)$$

In some cases, the solution of the problem for an insoluble impurity can be used as an approximate solution of the problem for a soluble impurity.

First of all, it is known under what conditions the impurity solubility effects are insignificant. We consider small perturbations of the film equilibrium. If the film thickness is small enough [8, 14]:

$$h \ll H, \quad H = \partial \Gamma / \partial c$$

a soluble impurity behaves as an insoluble one and the transfer equation (1.8) has a small error: $h/H \ll 1$ ($c$ is the concentration on the surface). The diffusion of the impurity affects sound waves in the film [14], increasing their attenuation. A decrease of $h/H$ leads to a reduction in the diffusion effect.

386