ANTIPLANE STRAIN OF A BODY
UNDERGOING LARGE-ROTATIONS

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Antiplane strain of a cylindrical elastic body undergoing large rotations under surface load in the absence of body loads is studied. The form of the elastic potential corresponding to this strain is found. The stresses, the strains, and the displacement are expressed in terms of pressure and two independent strains and the pressure is expressed in terms of the linear strain invariant. For the strains and displacement, nonlinear boundary-value problems are formulated and their ellipticity conditions are given. The linear problem for the displacement is obtained by transformation of variables. An example of determining the displacement is considered.

Key words: displacement, Almansi strains, rotations, Cauchy stresses, elastic potential, nonlinearity, boundary-value problem.

In a number of cases, rotations of elements of a deformed body can exceed substantially elongations and shears. This situation occurs, in particular, for deformation of flexible bodies and also massive bodies near the external and internal boundaries. For these cases, the strain–displacement relations which occupy intermediate position between the formulas of linear elasticity and the general nonlinear relations were obtained in [1]. Using these relations, we study the antiplane strain of an isotropic cylindrical body in the context of the nonlinear theory of elasticity in the actual variables \(x_1, x_2, x_3\) (\(x_1 = x\) and \(x_2 = y\) are the transverse coordinates and \(x_3 = z\) is the longitudinal coordinate) assuming that body forces are absent and surface load is given.

This model is determined by equilibrium equations, Murnaghan’s law, compatibility equation, relation of the strain invariants in terms of its components, and strain–displacement relations [2]. We write these relations in actual variables.

Expressing the displacement gradients \(\partial_k u_l\) in terms of the symmetric component \(e_{kl}\) and asymmetric component \(\omega_{kl}\):

\[
\partial_k u_l = e_{kl} + \omega_{kl} \quad (\partial_k = \partial/\partial x_k),
\]

\[
2e_{kl} = \partial_k u_l + \partial_l u_k, \quad 2\omega_{kl} = \partial_k u_l - \partial_l u_k
\]

[\(e_{kl}\) are components of the linear strain tensor (elongations and shears) and \(\omega_{kl}\) are the rotation-tensor components], we write Novozhilov’s formulas for the Almansi strains \(E_{kl}\) as

\[
2E_{kl} = 2e_{kl} - \omega_{km}\omega_{lm}
\]

[the right side (2) contains terms of the same order of magnitude]. In Eqs. (1) and (2) and below the subscripts take the values 1, 2, and 3; summation is performed over repeated indices.

For the antiplane strain of a cylindrical body (displacement is directed along the body and does not depend on the longitudinal coordinate \([3, 4]\)), we obtain

\[
u_1 = 0, \quad u_2 = 0, \quad u_3 = w(x, y).
\]
In accordance with (1), we have
\[ e_{11} = e_{22} = e_{33} = e_{12} = 0, \quad e_{31} = \partial_x w / 2, \quad e_{32} = \partial_y w / 2, \]
\[ \omega_{11} = \omega_{22} = \omega_{33} = \omega_{12} = 0, \quad \omega_{31} = -\partial_x w / 2, \quad \omega_{32} = \partial_y w / 2 \]
and, hence, formulas (2) yield
\[ E_{11} = - (\partial_x w)^2 / 8, \quad E_{22} = - (\partial_y w)^2 / 8, \quad E_{33} = ((\partial_x w)^2 + (\partial_y w)^2) / 8, \]
\[ E_{12} = - \partial_x w \partial_y w / 8, \quad E_{31} = \partial_x w / 2, \quad E_{32} = \partial_y w / 2, \quad E_{kl} = E_{kl}(x, y). \] (3)
Eliminating the displacement from (3), we obtain the finite and differential strain-compatibility conditions
\[ E_{11} = - E_{31}^2 / 2, \quad E_{22} = - E_{32}^2 / 2, \quad E_{33} = -(E_{31}^2 + E_{32}^2) / 2, \quad E_{12} = - E_{31} E_{32} / 2, \] (4)
The finite conditions allow one to express the strains in terms of the two independent components \( E_{31} \) and \( E_{32} \), whereas the differential condition establishes a differential relation between them.

According to the equalities \( 2E_{31} = \partial_x w \) and \( 2E_{32} = \partial_y w \) in (3), the independent strains determine the displacement by quadrature
\[ w = 2 \int_{(x_0, y_0)}^{(x, y)} (E_{31} \, dx + E_{32} \, dy) + w_0 \quad (w_0 = \text{const}), \] (5)
in which, according to (4), the integral is path independent and the constant is the displacement specified at the boundary point.

By virtue of (4), the basic strain invariants \( E_k \) as functions of the strain-tensor components or functions of two independent strains are determined by the formulas
\[ E_1 = E = E_{kk} = -(E_{31}^2 + E_{32}^2), \]
\[ 2E_2 = E_{kk} E_{ll} - E_{kl} E_{lk} = -2(E_{31}^2 + E_{32}^2)(1 - (E_{31}^2 + E_{32}^2) / 4), \quad E_3 = \det E_{kl} = 0. \]
These relations imply the properties of the invariants
\[ 4E_2 = E(4 + E), \quad E_3 = 0, \quad 1 - 2E_1 + 4E_2 - 8E_3 = (1 + E)^2, \quad E_k = E_k(x, y), \] (6)
i.e., the invariants are constant along the body and expressed in terms of the linear invariant.

For an isotropic body, the elastic potential \( U \) and the material density \( \rho \) are functions of the basic strain invariants:
\[ U = U(E_1, E_2, E_3), \quad \rho = \rho_0(1 - 2E_1 + 4E_2 - 8E_3)^{1/2} \]
(\( \rho_0 \) is the initial density). For the antiplane strain, by virtue of (6), these quantities depend only on the linear invariant:
\[ U = U(E), \quad \rho = \rho_0(1 + E). \] (7)
It follows that the body is compressible for the Novozhilov nonlinear model, whereas it exhibits incompressible behavior for this strain in the general geometrically nonlinear case [5].

Using Murnaghan’s law
\[ P_{kl} = \frac{\rho}{\rho_0} (\delta_{kn} - 2E_{kn}) \frac{\partial U}{\partial E_{ln}}, \]
(\( \delta_{kn} \) is the Kronecker symbol), which relates the Cauchy stress \( P_{kl} \) to the Almansi strain \( E_{kl} \) and taking into account (7) and the relations
\[ E = E_{ln} \delta_{nl}, \quad \frac{\partial E}{\partial E_{ln}} = \delta_{nl}, \quad \frac{\partial U}{\partial E_{ln}} = \frac{\partial U}{\partial E} \frac{\partial E}{\partial E_{ln}} = U'(E) \delta_{nl} \]
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