DISPERSION AND BLOCKAGE EFFECTS IN THE FLOW OVER A SILL

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The problem of a homogeneous heavy liquid flow over a local obstacle is considered in the long-wave approximation. The steady and unsteady waves in the vicinity of the obstacle are described by second-order models of the shallow-water theory and their hyperbolic approximations. The flow in the vicinity of the leading and trailing edges of bluff bodies (sills and steps) is studied. The solution of the problem of the blocked zone upstream of the step is constructed.

Key words: homogeneous liquid, equations of the shallow-water theory, dispersion effects, flow over a sill.

INTRODUCTION

The second-order approximation of the shallow water theory is used to model interaction of nonlinear waves. There are various versions of equations to take into account the influence of dispersion effects on the structure of nonlinear surface and internal waves in the long-wave approximation [1–5]. Alternative formulations of the models have been recently proposed to describe these effects in flows of a homogeneous liquid with a free boundary. Such models describe the evolution of nonlinear dispersion waves within the framework of hyperbolic systems of equations of the shallow water theory [4]. The effect of non-hydrostatic distributions of pressure is taken into account in these models by using additional “internal” variables in the equations. A preliminary analysis shows that hyperbolic dispersion models can be used to describe the evolution of surface waves above an uneven bottom, in addition to more widespread non-hyperbolic models corresponding to the second-order approximation of the shallow water theory. The main advantage of dispersive hyperbolic equations of the shallow water theory is substantial simplification of the algorithms of numerical calculation of multidimensional unsteady flows and of formulation of the boundary conditions, in particular, near the coast line.

Another class of problems for which dispersive hyperbolic equations provide a more explicit formulation of the problem than models of the second-order approximation is associated with formulation of conditions for upstream control of the flow through insertion of a local obstacle into the channel. Conditions of this type in open-channel hydraulics were determined for the classical shallow water equations. The local obstacle controls the flow if a steady-state transcritical flow is formed in the vicinity of the obstacle. For flows in a horizontal channel, the condition of providing a critical flow above the wave crest determines the relation between the mass flow rate and the flow depth immediately ahead of the obstacle and is independent of the shape of the latter. For the second-order approximation, the condition of the critical flow is replaced by more complicated conditions of existence of a regular steady-state flow in the vicinity of a local constriction of the channel and do not allow the corresponding generalization in analyzing unsteady problems. Therefore, a hyperbolic analog of equations of the second-order approximation of the shallow water theory can be effectively used to formulate the boundary conditions in the problem of a liquid flow in a finite-length channel.
Steady-state and unsteady problems of a plane-parallel flow of a homogeneous heavy liquid above a local obstacle are considered in the present paper. Models developed by Green and Naghdi [2, 3], by Serre [6], and hyperbolic approximations of these models are used to analyze the wave structure near the obstacle. The flows in the vicinity of the leading and trailing edges of bluff bodies (sill and step) are studied; the solution of the problem of a blocked zone upstream of a step is constructed.

1. MATHEMATICAL MODELS

1.1. Dispersive Hyperbolic Models. The following system of equations is used to describe nonlinear dispersion waves in continuous media [1]:

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} = 0,
\]

\[
\frac{\partial (\rho u)}{\partial t} + \frac{\partial (\rho u^2 + P)}{\partial x} = 0.
\]

Here \(\rho\) is the medium density and \(u\) is the mean velocity.

In contrast to gas-dynamic equations, which describe barotropic flows, the pressure \(P\) is assumed to depend not only on density but also on its material derivatives:

\[
P = P\left(\rho, \frac{d\rho}{dt}, \frac{d^2\rho}{dt^2}\right), \quad \frac{d}{dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x}.
\]

The class of equations considered includes the equations of bubbly liquids, equations of the shallow water theory, etc. [1–3]. These equations are used to describe media with “internal inertia,” i.e., heterogeneous media where a certain part of the total energy transforms to the energy of small-scale motions, which produces a significant effect on the wave structure of the flows. Numerical implementation of one-dimensional models and, moreover, multidimensional analogs of Eqs. (1) and (2) involves some difficulties in imposing the boundary conditions (e.g., in the problem of waves rolling onto the shore), which are caused by non-hyperbolicity of the system considered.

These difficulties can be partly resolved by constructing the hyperbolic approximation (1). Systems of heterogeneous hyperbolic equations, which occupy an intermediate position between the first- and second-order approximations, were derived in [4] for the second-order approximation of the shallow water theory. A hyperbolic model is obtained by additional averaging of the equations of the second-order approximation and by introducing new “internal” variables. The scale of averaging is assumed to be rather small, which allows the values of the variables \(\rho\) and \(u\) to be replaced in the equations by their mean values. In calculating the total pressure, however, derivatives from the “instantaneous” values of density \(\tilde{\rho}\) and velocity \(\tilde{u}\) are used. As applied to Eq. (1), this means that the pressure depends on the averaged density \(\rho\) and material derivatives of \(\tilde{\rho}\):

\[
P = P\left(\rho, \frac{d\tilde{\rho}}{dt}, \frac{d^2\tilde{\rho}}{dt^2}\right), \quad \frac{d}{dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x}.
\]

The relation between the averaged and “internal” variables is found by expanding the functions \(\tilde{\rho}(s, \xi) = \tilde{\rho}(s)\) into a Taylor series along the trajectory \(x = x(s, \xi)\) (\(\xi\) is a fixed Lagrangian coordinate of the particle):

\[
\tilde{\rho}(s) = \tilde{\rho}(t) + \tilde{\rho}'(t)(s - t) + \tilde{\rho}''(t)(s - t)^2/2 + O(\tau^2), \quad s \in (t - \tau, t + \tau).
\]

By virtue of Eq. (3), the mean value \(\rho(t)\) is related to the instantaneous value \(\tilde{\rho}(t)\) by the expression

\[
\rho(t) = \frac{1}{2\tau} \int_{t-\tau}^{t+\tau} \tilde{\rho}(s) \, ds = \tilde{\rho}(t) + \frac{1}{6} \tilde{\rho}''(t)\tau^2 + O(\tau^2),
\]

which can be presented in the form

\[
\tilde{\rho}''(t) = \alpha(\rho(t) - \tilde{\rho}(t)) + O(\tau), \quad \alpha = 6/\tau^2.
\]