OPTIMAL CONTROL OF MHD-FLOW DECELERATION

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A problem of pulsed control for a three-dimensional magnetohydrodynamic (MHD) model is considered. It is demonstrated that singularities of the solution of MHD equations do not develop with time because they are suppressed by a magnetic field. The existence of an optimal control is proved. An optimality system with the solution regular in time as a whole is constructed.

Key words: magnetohydrodynamic equations, pulsed control, optimality conditions.

Introduction. A flow of a homogeneous viscous incompressible conducting fluid in a bounded, simply connected domain $\Omega \subset \mathbb{R}^3$ with a connected boundary $\Gamma = \partial \Omega$ is modeled by magnetohydrodynamic (MHD) equations in dimensionless variables:
\begin{align*}
\text{div } u &= 0, \quad \text{div } B = 0; \\
\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u &= -\nabla p + S \cdot \text{rot } B \times B + q, \quad x \in \Omega, \quad t \in (0, T); \\
\frac{\partial B}{\partial t} + \text{rot } E &= 0, \quad \text{rot } B = \frac{1}{\nu_m} (E + u \times B).
\end{align*}

Here $u$, $B$, and $E$ are the vector fields of velocity, magnetic induction, and electric strength, respectively, $p$ is the pressure, $q = q(x)$ is the density of external forces, $\nu = 1/\text{Re}$, $\nu_m = 1/\text{Re}_m$, $S = M^2/(\text{Re} \text{Re}_m)$, $\text{Re}$ is the Reynolds numbers, $\text{Re}_m$ is the magnetic Reynolds number, and $M$ is the Hartmann number.

Equations (1)–(3) are supplemented by the conditions on the boundary $\Gamma$ of the flow domain
\begin{align*}
u u &= 0, \quad B \cdot n = 0, \quad n \times E = 0, \quad (x, t) \in \Gamma \times (0, T) \\
(u|_{t=0} &= u_0(x), \quad B|_{t=0} = 0, \quad x \in \Omega.
\end{align*}

A method of flow deceleration with the use of pulsed control by a magnetic field is proposed. The control functions are chosen to be the jumps $b_i$ of the magnetic field at the times $0 < t_1 < t_2 < \ldots < t_m < T$. In this case, the MHD flow is described by Eqs. (1) and (3) and by the equations
\begin{align*}
\frac{\partial B}{\partial t} + \text{rot } E &= \sum_{i=1}^{m} \delta(t - t_i)b_i, \quad \text{rot } B = \frac{1}{\nu_m} (E + u \times B)
\end{align*}

with the initial-boundary conditions (4) and (5). Here $\delta(t - t_i)$ is the Dirac $\delta$-function with the support at the point $t_i$.

The task is to minimize the functional
\begin{align*}
J &= \frac{1}{4} \int_{0}^{T} \int_{\Omega} \left( (\text{rot } u)^2 + (\text{rot } B)^2 \right) \, dx \, dt + \frac{\lambda}{2} \sum_{i=1}^{m} \int_{\Omega} (\text{rot } b_i)^2 \, dx
\end{align*}

($\lambda > 0$ is the regularization parameter).
The classical boundary-value problems for the evolutionary model (1)–(3) were studied in [1, 2]. The problems of optimal control of the evolutionary Navier–Stokes systems were first studied in [3–5]. The optimal control of unsteady MHD equations was considered in [6]. In studying the problems of optimal control of three-dimensional systems of Navier–Stokes-type equations, the main problem is to ensure regularity of the optimal state of the flow. For the statement examined, it is demonstrated that singularities of the solution (in the Leray sense) do not develop with time because they are suppressed by a magnetic field. Solvability of the control problem is proved. An optimality system is constructed, and its regularity in time as a whole is justified. The method of deriving the optimality conditions is close to the method proposed in [7].

1. Formalization and Solvability of the Control Problem. To simplify transformations, we use the re-normalization

\[ B = \sqrt{S} B, \quad E = \sqrt{S} E. \]

Then, system (1)–(3) acquires the form

\[ \text{div}\ u = 0, \quad \text{div}\ B = 0; \]

\[ u' - \nu \Delta u + (u \cdot \nabla)u = -\nabla p + \text{rot}\ B \times B + q, \quad x \in \Omega, \quad t \in (0, T); \]

\[ B' + \text{rot}\ E = 0, \quad E = \nu_m \text{rot}\ B - u \times B, \]

where \( u' = \partial u/\partial t \) and \( B' = \partial B/\partial t \).

We consider the spaces of vector functions and operators necessary to analyze the control problem [2]. Let \( \Omega \) be a simply connected domain in the space \( \mathbb{R}^3 \) with a connected boundary \( \Gamma \in C^2 \). We introduce the spaces

\[ U_1 = \{ v \in C^\infty(\Omega): \text{div}\ v = 0, \ x \in \Omega, \ v = 0, \ x \in \Gamma \}, \]

\[ U_2 = \{ v \in C^\infty(\bar{\Omega}): \text{div}\ v = 0, \ x \in \Omega, \ n \cdot v = 0, \ x \in \Gamma \}, \]

\( V_1 \) is the closure of \( U_1 \) with the norm \( W^1_2(\Omega) \), \( V_2 \) is the closure of \( U_2 \) with the norm \( W^2_1(\Omega) \), \( H_1 \) is the closure of \( U_1 \) with the norm \( L^2(\Omega) \), and \( H_2 \) is the closure of \( U_2 \) with the norm \( L^2(\Omega) \).

We assume that

\[ (u, v)_0 = \int_\Omega (u \cdot v) \, dx \]

is a scalar product in the spaces \( H_1 \) and \( H_2 \),

\[ ((u, v)) = (\text{rot}\ u, \text{rot}\ v)_0 = \int_\Omega (\text{rot}\ u \cdot \text{rot}\ v) \, dx \quad \forall u, v \in V_1, V_2 \]

is a scalar product in the spaces \( V_1 \) and \( V_2 \), and the norm determined by this product is equivalent to the space norm \( W^1_2(\Omega) \). Let \( X \) be a Banach space. Then, \( L^p(0, T; X) \) \( C(\{0, T\}; X) \) designates a space \( L^p \) (class \( C \)) of functions determined on the interval \([0, T]\) with their values in the space \( X \). We determine the spaces

\[ V = V_1 \times V_2, \quad H = H_1 \times H_2, \quad V \subset H = H' \subset V'. \]

This embedding of spaces is dense and continuous. The norms in the spaces \( V \) and \( H \) and in the adjoint space \( V' \) are denoted by \( |||\cdot|||_V \), \( ||\cdot||_H \), and \( ||\cdot||_V' \), respectively; \((\cdot, \cdot)\) is the duality relation between \( V' \) and \( V \) and a scalar product in \( H \):

\[ (y, z) = (u, v)_0 + (B, w)_0, \quad (y, z)_V = ((u, v)) + ((B, w)) \quad \forall y = \{u, B\}, \ z = \{v, w\}. \]

We introduce the mappings \( A_1: V_1 \to V'_1 \), \( A_2: V_2 \to V'_2 \), \( A: V \to V' \), and \( B: V \times V \to V' \), using the relations

\[ (Ay, z) = \nu((u, v)) + \nu_m((B, w)) = \nu(A_1 u, v) + \nu_m(A_2 B, w), \]

\[ b(u, v, w) = \int_\Omega (u \cdot \nabla) v \cdot w \, dx, \]

\[ (B(y_1, y_2), z) = b(u_1, u_2, v) - b(B_1, B_2, v) + b(B_1, B_2, w) - b(B_1, u_2, w), \]

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